On bounding the difference between the maximum degree and the chromatic number by a constant

Vera Weila∗, Oliver Schaudtb

aChair of Management Science, RWTH Aachen University, Kackertstr. 7, 52072 Aachen, Germany
bDepartment for Computer Science, University of Cologne, Weyertal 80, 50931 Cologne, Germany

Abstract

For every \( k \in \mathbb{N}_0 \), we consider graphs in which for any induced subgraph, \( \Delta \leq \chi - 1 + k \) holds, where \( \Delta \) is the maximum degree and \( \chi \) is the chromatic number of the subgraph. Let us call this family of graphs \( \Upsilon_k \). We give a finite forbidden induced subgraph characterization of \( \Upsilon_k \) for every \( k \).

We compare these results with those given in [6], where we studied the graphs in which for any induced subgraph \( \Delta \leq \omega - 1 + k \) holds, where \( \omega \) denotes the clique number of a graph.

In particular, we introduce the class of neighborhood perfect graphs, that is, those graphs where the neighborhood of every vertex is perfect. We find a nice characterization of this graph class in terms of \( \Omega_k \) and \( \Upsilon_k \): We prove that a graph \( G \) is a neighborhood perfect graph if and only if for every induced subgraph \( H \) of \( G \), \( H \in \Upsilon_k \) if and only if \( H \in \Omega_k \) for all \( k \in \mathbb{N}_0 \).

Keywords: maximum degree, graph coloring, chromatic number, structural characterization of families of graphs, hereditary graph class, neighborhood perfect graphs.

1. Introduction

A graph class \( \mathcal{G} \) is called hereditary if for every graph \( G \in \mathcal{G} \), every induced subgraph of \( G \) is also a member of \( \mathcal{G} \). If we describe a graph class \( \mathcal{G} \) by excluding a (not necessarily finite) set of graphs as induced subgraphs, then this graph class is hereditary.

Among the best studied hereditary graph classes is the class of perfect graphs. In this context, recall that a clique in a graph is a set of vertices of the graph that are pairwise adjacent. A maximal clique that is of largest size in a graph \( G \) is called a maximum clique of the graph. By \( \omega(G) \) we denote the largest size of a maximum clique in a graph \( G \). Moreover, we call an assignment of colors to every vertex of the graph such that adjacent vertices do not receive the same color a coloring of a graph. A coloring that uses a minimum number of colors is called an optimal coloring. The number of colors used in an optimal coloring

∗Corresponding author

Email addresses: vera.weil@oms.rwth-aachen.de (Vera Weil), schaudto@uni-koeln.de (Oliver Schaudt)
of a graph $G$ is denoted by $\chi(G)$, the so called chromatic number. Now, if $G$ is a perfect graph, then for $G$ and all its induced subgraphs the clique number and the chromatic number coincide. By the Strong Perfect Graph Theorem [2], those graphs can be explicitly described by a set of forbidden induced subgraphs, namely the set of odd cycles of length at least five, and their complements, also called odd holes and odd anti-holes. The class of perfect graphs is of great interest from both a structural and an algorithmic point of view (cf. [3, 4]).

Given the chromatic number $\chi$ and the maximum degree $\Delta$ of a graph, by Brook’s Theorem [5], $\chi \leq \Delta + 1$ holds. On the other hand, it is not possible to give a lower bound on $\chi$ in terms of $\Delta$ only. The infinite set of complete bipartite graphs $K_{1,p}$, $p \in \mathbb{N}$, yields an example for a family of graphs where the difference between $\chi$ and $\Delta$ is unbounded.

If we require for a graph that the difference between $\Delta$ and $\chi$ is bounded, then this does not imply that this difference is bounded for all its induced subgraphs, too. Consider, for example, for a given $p \in \mathbb{N}$, a $K_{1,p}$, choose one vertex, say $v$, in the $p$-partition and attach every vertex of a clique of size $p - 1$ to $v$. In the resulting graph, $\Delta$ and $\chi$ equal $p$, hence the difference equals 0, but the induced $K_{1,p}$ yields a graph where the difference is $p - 2$.

This gives rise to the question which graphs guarantee that, in every induced subgraph, the difference between the maximum degree and the chromatic number is at most some given number $k$?

We answer the above question in the following way. For every $k \in \mathbb{N}_0$, let $T_k$ be the class of graphs $G$ for which $\Delta(H) + 1 \leq \chi(H) + k$ holds for all induced subgraphs $H$ of $G$. For every $k$, we describe all graphs contained in $T_k$ by giving a minimal forbidden induced subgraph characterization. Moreover, we are able to prove that the order of the respective minimal forbidden induced subgraph set is finite. This gives that the problem of recognition of such graphs can be solved in polynomial time.

An equally interesting question is the following. Is it possible to characterize those graphs in which the difference between the maximum degree and the clique number is bounded by a constant, for all induced subgraphs of the graph? Or, to be more precise, let $G$ be a graph. Then $G \in \Omega_k$, $k \in \mathbb{N}_0$, if for all induced subgraphs $H$ of $G$, $\Delta(H) \leq \omega(H) + k$, where $\omega(H)$ denotes the maximum clique size, also known as the clique number, of $H$. Is it possible to characterize $\Omega_k$ by a finite set of minimal forbidden subgraphs? A positive answer to this question is given in [6]. There, we also give some further results on this family of graphs. Below, we refer to these results for $\Omega_k$ and find some interesting relations between these graph families and $T_k$. Roughly speaking, we ask when $T_k$ and $\Omega_k$ describe the same graph set, and dedicate Section 3 to this question.

Note that we have to distinguish between induced subgraphs and (partial) subgraphs. Since we deal with graph invariants, we are allowed to treat isomorphic graphs as identical. For example, if a graph $G$ is an induced subgraph of a graph $H$ and $G$ is isomorphic to a graph $L$, then we say that $L$ is an induced subgraph of $H$.

Let $F(\chi, k)$ denote the set of minimal forbidden induced subgraphs of $T_k$. Hence for every graph $F \in F(\chi, k)$, $F \not\in T_k$ and all proper induced subgraphs of $F$ are contained in $T_k$. Observe that $G \in T_k$ if and only if $G$ is $F(\chi, k)$-free.
2. Bounding the difference between $\Delta$ and $\chi$ by a constant, for all induced subgraphs

Recall from the introduction that for a fixed $k \in \mathbb{N}_0$, $\mathcal{T}_k$ contains all graphs $G$ such that for all induced subgraphs of $G$, the difference between the maximum degree $\Delta$ and the chromatic number $\chi$ of the induced subgraph is bounded by $k - 1$. The set of minimal forbidden induced subgraphs of $\mathcal{T}_k$ is denoted by $F(\chi, k)$. With $V(G)$, we denote the vertex set of a graph $G$. All vertices adjacent to a vertex $v \in V(G)$ form the neighborhood of $v$, denoted by $N(v)$. The degree of $v$ corresponds to $|N(v)|$. A vertex is dominating in a graph if it is adjacent to all other vertices of the graph. In a coloring of a graph, a color class is the set of all vertices to which the same color is assigned to.

Our results are primarily based on Theorem 1, which characterizes the minimal forbidden induced subgraphs of $\mathcal{T}_k$ by three properties.

**Theorem 1.** Let $G$ be a graph. $G \in F(\chi, k)$ if and only if the following conditions hold:

1. $G$ has a unique dominating vertex $v$,
2. each color class in every optimal coloring of $G - v$ consists of at least two vertices,
3. $\Delta(G) = \chi(G) + k$.

In particular, $\Delta(G) = |V(G)| - 1$ and $\chi(G) = |V(G)| - k - 1$.

**Proof.** Let $G \in F(\chi, k)$. We show that the three conditions hold. Note that since $G$ is a minimal forbidden induced subgraph, all induced subgraphs of $G$ are contained in $\mathcal{T}_k$ except for $G$ itself. Thus $\Delta(G) \geq \chi(G) + k - 1$.

Choose a vertex $v$ of maximum degree in $G$ and let $H$ be the graph induced in $G$ by the vertex set $\{v\} \cup N(v)$. Observe that $H \subseteq G$ is not in $\mathcal{T}_k$, since $\Delta(H) = \Delta(G) > \chi(G) + k - 1 \geq \chi(H) + k - 1$. Hence, $H \cong G$ by minimality of $G$, and thus, $G$ contains a dominating vertex, namely $v$.

Assume there exists $x \in V(G) \setminus \{v\}$ such that $\chi(G - x) = \chi(G) - 1$. Then

$$\Delta(G - x) = \Delta(G) - 1 \geq \chi(G) + k - 1 = \chi(G - x) + k.$$  \hspace{1cm} (1)

Thus $G - x \notin \mathcal{T}_k$, contradicting the minimality of $G$. Hence $v$ is a unique dominating vertex of $G$ and thus, Condition 1 holds. Moreover, (1) implies Condition 2.

Let $x \in N(v)$. Due to Condition 1, the degree of $x$ is at most $\Delta(G) - 2$, and hence, $\Delta(G - x) = \Delta(G) - 1$. Due to Condition 2, $\chi(G - x) = \chi(G)$. Assume $\Delta(G) \geq \chi(G) + k + 1$. Then

$$\Delta(G - x) = \Delta(G) - 1 \geq (\chi(G) + k + 1) - 1 = \chi(G - x) + k.$$

That is, $G - x \notin \mathcal{T}_k$, a contradiction. Hence $\Delta(G) = \chi(G) + k$, and the third condition follows.

On the other hand, let $G$ obey Conditions 1, 2 and 3. We have to prove that $G \in F(\chi, k)$. Since

$$\Delta(G) = \chi(G) + k > \chi(G) + k - 1,$$

$G$ is a forbidden induced subgraph for every graph contained in $\mathcal{T}_k$. To see that $G$ is minimal, assume the opposite. Let $L$ be a minimal forbidden induced
subgraph that is an induced subgraph of $G$, hence $L \in F(\chi, k)$. By assumption, $G \neq L$. We already proved that $L$ has the following properties: $L$ has a unique dominating vertex $y$, each color class in every optimal coloring of $L$ consists of at least two vertices, and $\Delta(L) = \chi(L) + k$. Let $S = V(G) - V(L)$ and recall that $\Delta(G) = V(G) - 1$. Thus,

$$\chi(L) + k = V(L) - 1 = V(G) - |S| - 1 = \Delta(G) - |S| = \chi(G) + k - |S|.$$ 

That is, $\chi(G) - \chi(L) = |S| = |V(G)| - |V(L)|$. In other words, every vertex contained in $V(G) \setminus V(L)$ coincides with its own color class in every optimal coloring of $G$, yielding $S = \{v\}$. This implies that $y$ is a further dominating vertex of $G$, contradicting Condition 1. This completes the proof of the characterization.

In particular, note that if Conditions 1, 2 and 3 hold for a graph $G$, then the dominating vertex $v$ has maximum degree, thus $\Delta(G) = |V(G)| - 1$. By Condition 3, $\Delta(G) = \chi(G) + k$, and therefore $\chi(G) = |V(G)| - k - 1$. \[\square\]

In [6], we introduced a family of hereditary graph classes that is quite similar to $\Upsilon_k$, $k \in \mathbb{N}_0$, namely the family of hereditary graph classes where the difference between $\Delta$ and $\omega$ is bounded by a constant. Precisely, let $\Omega_j$, $j \in \mathbb{N}_0$, be the set of graphs $G$ where every induced subgraph $H$ of $G$, including $G$ itself, obeys $\Delta(H) \leq \omega(H) + j - 1$. Let $F(\omega, j)$ denote the set of minimal forbidden induced subgraphs of $\Omega_j$. In [6], we give a characterization of $F(\omega, j)$ (see Theorem 2) where the analogism to Theorem 1 is easy to see.

**Theorem 2 ([6]).** Let $G$ be a graph. $G \in F(\omega, k)$ if and only if the following conditions hold:

1. $G$ has a unique dominating vertex $v$,
2. the intersection of all maximum cliques of $G$ contains solely $v$,
3. $\Delta(G) = \omega(G) + k$.

In particular, $\Delta(G) = |V(G)| - 1$ and $\omega(G) = |V(G)| - k - 1$.

Our next result, Proposition 1, provides a bound in terms of $k$ on the order of the minimal forbidden induced subgraphs of $\Upsilon_k$. For $n \in \mathbb{N}$, $K_n$ is the complete graph on $n$ vertices.

**Proposition 1.** Let $G \in F(\chi, k)$. Then $2\chi(G) - 2 \leq \Delta(G) \leq 2k + 2$. Moreover, $2 \leq \chi(G) \leq k + 2$.

**Proof.** Let $G \in F(\chi, k)$. Recall that by Theorem 1, Condition 2, except for the dominating vertex, every vertex is in a color class that contains at least two vertices, given an optimal coloring. Thus,

$$\Delta(G) \geq 2(\chi(G) - 1). \quad (2)$$

To show the upper bound on $\Delta(G)$, recall that by Theorem 1, it holds that $\chi(G) + k = \Delta(G)$. Further, observe that by (2),

$$\chi(G) + k = \Delta(G) \geq 2(\chi(G) - 1).$$

Hence, $\chi(G) \leq k + 2$ and therefore $\Delta(G) \leq 2k + 2$. Finally, since $K_1 \in \Omega_k$ for all $k \geq 0$, $\chi(G) \geq 2$. This completes the proof. \[\square\]
Proposition 1 has an important consequence: it yields a bound for the order of minimal forbidden induced subgraphs. Hence, for any fixed \( k \in \mathbb{N}_0 \), \( F(\chi, k) \) is a subset of the set of graphs that have at most \( 2k + 3 \) vertices, and therefore is finite. Hence, the characterization given in Theorem 1 leads to one of our central results:

**Observation 1.** For every \( k \in \mathbb{N}_0 \), the set of minimal forbidden induced subgraphs of \( T_k \) is finite.

This gives that the problem of recognition of these graphs can be solved in polynomial time.

Note that for any fixed \( j \in \mathbb{N}_0 \), the set \( F(\omega, j) \) is also finite (cf. [6]). More similarities become clear when comparing the sets \( \Omega_k \) and \( \Upsilon_k \). For example, every graph in which hereditarily the maximum degree is bounded by the clique number plus a constant is also a graph in which, hereditarily, the maximum degree is bounded by the chromatic number plus the same constant. This follows directly from the fact that the chromatic number is always at least as large as the clique number. Hence we can state the following observation.

**Observation 2.** For every \( k \in \mathbb{N}_0 \), \( \Omega_k \subseteq \Upsilon_k \).

Note that Observation 2 does not necessarily imply \( F(\chi, k) \subseteq F(\omega, k) \). Every graph contained in \( F(\chi, k) \) is forbidden as a subgraph of a graph in \( \Omega_k \), but with regard to this property not necessarily minimal.

However, if a graph in \( F(\chi, k) \) is perfect, it is also contained in \( F(\omega, k) \), as demonstrated by the next results. In a graph \( G \), we say that a vertex \( v \) is a \( \phi \)-critical vertex if \( \phi(G - v) < \phi(G) \), for \( \phi \in \{\omega, \chi\} \). Recall that a graph is perfect if for all induced subgraphs of the graph, the clique number and the chromatic number coincide.

**Lemma 1.** Let \( G \) be a perfect graph. Then the intersection of all maximum cliques of \( G \) is empty if and only if in every optimal coloring of \( G \), every color class consists of at least 2 vertices.

**Proof.** Let \( G \) be a perfect graph. Note that a \( \omega \)-critical vertex is \( \chi \)-critical in \( G \), and vice versa. Moreover, a \( \omega \)-critical vertex is contained in all maximum cliques of \( G \) and a vertex is \( \chi \)-critical if and only if there exists an optimal coloring of \( G \) such that \( v \) forms its own color class. Hence, the intersection of all maximum cliques of \( G \) is empty if and only if the set of \( \omega \)-critical vertices of \( G \) is empty if and only if the set of \( \chi \)-critical vertices in \( G \) is empty. This is the case if and only if in every optimal coloring of \( G \), every color class consists of at least 2 vertices. \( \Box \)

With this lemma, we can now formulate the following result.

**Theorem 3.** Let \( k \in \mathbb{N}_0 \) and let \( G \) be a perfect graph. Then \( G \in F(\omega, k) \) if and only if \( G \in F(\chi, k) \). In particular, let \( PG \) denote the class of perfect graphs. Then \( F(\omega, k) \cap PG = F(\chi, k) \cap PG \).

Moreover, if all graphs in \( F(\omega, k) \) are perfect, then \( F(\omega, k) = F(\chi, k) \).

**Proof.** Let \( k \in \mathbb{N}_0 \) and let \( G \) be a perfect graph. Observe that \( G \in F(\omega, k) \) if and only if \( G \) meets Conditions 1, 2 and 3 of Theorem 2. Obviously, Condition 1 of Theorem 2 and Condition 1 of Theorem 1 coincide. By Lemma 1, \( G \) obeys
Condition 2 of Theorem 2 if and only if $G$ obeys Condition 2 of Theorem 1, since $G$ is perfect. Finally, $\Delta(G) = \omega(G) + k$ is equivalent to $\Delta(G) = \chi(G) + 1$, again due to perfectness of $G$. All in all, $G \in F(\omega, k)$, and only if $G \in F(\chi, k)$.

Let $F(\omega, k)$ be a subset of the set of perfect graphs. By the previous result, $F(\omega, k) \subseteq F(\chi, k)$. Let $H \in F(\chi, k)$. If $\omega(H) < \chi(H)$, then $\Delta(H) = \chi(H) + k > \omega(H) + k$. Hence $H$ is a forbidden subgraph for $\Omega_k$. Thus, $H$ must contain a graph $F \in F(\omega, k)$ as induced subgraph. Since $F$ is perfect, $F \in F(\chi, k)$, and thus $F = H$.

If $\omega(H) = \chi(H)$, then $\Delta(H) = \omega(H) + k = \chi(H) + k$. But then, $H$ obeys Conditions 1 and 3 of Theorem 2. Hence, there exists a vertex $w \in V(H)$ that is not the dominating vertex and that is contained in the intersection of all maximum cliques of $H$. Then $\chi(H - w) = \chi(H)$ and $\omega(H - w) = \omega(H) - 1$. Thus, $\Delta(H - w) = \omega(H - w) + k + 1$; therefore, $H - w$ is a forbidden subgraph for $\Omega_k$. In particular, $H - w$ contains as a subgraph a graph in $F(\omega, k)$, say $F$. By presumption, $F$ is a perfect graph and hence, $F \in F(\chi, k)$. In other words, $F = H$. That is, in both cases, $H \in F(\chi, k)$ implies $H \in F(\omega, k)$.

With Theorem 3, it is easy to show that the sets $\Omega_k$ and $\Upsilon_k$ are the same for $k = 0$ and $k = 1$. For $s \in \mathbb{N}$, let $P_s$ denote the path on $s$ vertices.

**Theorem 4.** $F(\chi, 0) = \{P_3\} = F(\omega, 0)$. That is, $\Upsilon_0$ consists of unions of complete graphs.

**Proof.** By Theorem 4 in [6], $F(\omega, 0) = P_3$. In particular, $F(\omega, 0)$ consists of perfect graphs only. Theorem 3 completes the proof.

Also for $k = 1$, $F(\omega, 1)$ and $F(\chi, 1)$ coincide. For $s \in \mathbb{N}$, let $C_s$ denote the cycle on $s$ vertices. For the graphs in the set $F(\chi, 1)$, cf. Figure 1.

![Figure 1: The graphs claw, gem, $W_4$, butterfly.](image)

**Theorem 5.** $F(\chi, 1) = \{\text{claw, gem, } W_4, \text{ butterfly}\} = F(\omega, 1)$.

**Proof.** By Theorem 5 in [6], $F(\omega, 1) = \{\text{claw, gem, } W_4, \text{ butterfly}\}$. Hence all graphs in $F(\omega, 1)$ are perfect graphs. Theorem 3 completes the proof.

The union of two graphs $G$ and $H$ is denoted by $G \cup H$. For $n, m \in \mathbb{N}$, $K_{n,m}$ denotes the complete bipartite graph where one partition consists of $n$ and the other of $m$ vertices. In order to compare the sets $F(\omega, 2)$ and $F(\chi, 2)$, we restate Theorem 6 of [6], in a slightly adapted version that is based on the observation that every $K_3$-free supergraph of $K_2 \cup K_2 \cup K_1$ on five vertices is either a subgraph of $K_{2,3}$ or is the $C_5$-graph. If we say subgraph respectively supergraph we allow both edges and vertices to be removed respectively added to the host graph.
Theorem 6 ([6]). Let $G$ be a graph. $G \in F(\omega, 2)$ if and only if $G$ contains a dominating vertex $v$ and one of the following holds:

1. $G - v \cong K_4$,
2. $G - v$ is a supergraph of $K_2 \cup K_2 \cup K_1$ and a subgraph of $K_{2,3}$,
3. $G - v \cong C_5$,
4. $G - v \cong S_3$,
5. $G - v$ is a supergraph of $K_3 \cup K_3$ and a subgraph of $K_6 - 3e$.

All graphs contained in $F(\chi, 2)$ are shown in Figure 2. With $C_5^{(3)}$ and $C_5^{(4)}$ we denote the graphs that correspond to a $C_5$ with a $K_1$ attached to three respectively four consecutive vertices of the $C_5$. Both graphs, drawn with a dominating vertex, can be found in Figure 2, namely the last two graphs in the second row.

Figure 2: The set $F(\chi, 2)$.

Theorem 7. Let $G$ be a graph. Then $G \in F(\chi, 2)$ if and only if $G$ contains a dominating vertex $v$ and one of the following holds:

1. $G - v \cong K_4$,
2. $G - v$ is a supergraph of $K_2 \cup K_2 \cup K_1$ and a subgraph of $K_{2,3}$,
3. $G - v$ consists of 6 vertices and is a subgraph of $K_6 - 3e$ such that one of the following holds:
   (a) $G - v \cong S_3$,
   (b) $G - v$ is a supergraph of $K_3 \cup K_3$,
   (c) $G - v \cong C_5^{(3)}$,
   (d) $G - v \cong C_5^{(4)}$. 

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In particular, $F(\omega, 2) \setminus \{W_5\} = F(\chi, 2) \setminus \{C_5^{(3)}, C_5^{(4)}\}$.

Proof. With Condition 1, 2 and 3, we refer to the conditions stated in Theorem 1. With [i.] we refer to Condition i listed in Theorem 7.

Let $G$ be a graph with a unique dominating vertex $v$. Let $G - v$ obey [1.], [2.], [3a.] or [3b.]. Then $G - v$ is a perfect graph and obeys Condition 1, 2, 4 or 5 of Theorem 6, respectively. By Theorem 3, $G \in F(\chi, 2)$. If $G - v$ obeys [3c.] or [3d.], then it is easy to see that $G - v$ obeys Condition 1, 2 and 3 of Theorem 1, and that hence, $G \in F(\chi, 2)$.

To show the reverse direction, let $G$ be a graph in $F(\chi, 2)$. Observe that by Condition 1 of Theorem 1, $G$ has a unique dominating vertex, say $v$. By Theorem 3, $F(\chi, 2) \cap PG = F(\omega, 2) \cap PG$. In other words, if $G$ is a graph in $F(\chi, 2)$ that is perfect, then and only then $G$ is a graph in $F(\omega, 2)$ that is perfect. Therefore, $G$ obeys [1.], [2.], [3a.] or [3b.]. Let now $G$ be a non-perfect graph.

By Proposition 1, we have $2 \leq \chi(G) \leq 4$, therefore $|G| = \Delta(G) + 1$ equals 5, 6 or 7 and thus $|G - v| = 4, 5$ or $6$ and $\chi(G - v) = 1, 2$ or 3, respectively. Non-perfectness implies $\chi(G - v) = 3$ and therefore $|G - v| = 6$. The only odd hole respectively anti-hole that can be embedded as induced subgraph in $G - v$ is therefore $C_5$. Hence, let $C$ be an induced $C_5$ in $G - v$ and let $u \in G - v$ be the vertex not in $C$. Since $v$ is a unique dominating vertex, $u$ is adjacent to at most four vertices of $C_5$. Moreover, if $u$ is adjacent to at most two vertices of $C_5$, or to three vertices of $C_5$ that are not consecutively ordered, then we always find a coloring of $G - v$ where one vertex forms a singleton color class, contradicting Condition 1 of Theorem 1. Hence, $G \cong C_5^{(3)}$ or $G \cong C_5^{(4)}$. By checking the three conditions listed in Theorem 1, is easy to see that both these graphs are in $F(\chi, 2)$. This completes the proof.

To sum up, $F(\omega, 0) = F(\chi, 0)$, $F(\omega, 1) = F(\chi, 1)$, but

$$F(\omega, 2) \setminus \{W_5\} = F(\chi, 2) \setminus \{C_5^{(3)}, C_5^{(4)}\}.$$ 

Observe that both $C_5^{(3)}$ and $C_5^{(4)}$ contain $W_5$ as induced subgraph. The question arises what separates the set $F(\chi, k)$ from the set $F(\omega, k)$ for a fixed $k \in \mathbb{N}$. In order to answer this question, we will generalize the result for $k = 2$ in the following section.

Before we proceed, observe that $|F(\chi, 0)| = 1$, $|F(\chi, 1)| = 4$ and $|F(\chi, 2)| = 24$. Moreover, $|F(\chi, 3)| = 402$ and $|F(\chi, 4)| = 25788$ (cf. [7]), hence, although finite, the sets of minimal forbidden induced subgraphs seem to grow very quickly compared to the increase of $k$. All minimal forbidden induced subgraphs for $k = 1, 2$ and 3 can be downloaded from House of Graphs [1] by searching for the keywords “maximum degree * chromatic number” or “chi(G) + k” where $k = 1, 2$ or 3.

3. Neighborhood perfect graphs

Recall that a graph is perfect if and only if it does not contain an odd hole or an odd anti-hole, by the Strong Perfect Graph Theorem [2]. We say that a graph is neighborhood perfect if in the graph every neighborhood of a vertex is perfect. It is easy to see that a graph is neighborhood perfect if and only if it does not contain the join of an odd hole with $K_1$, that is, an odd wheel, and the join of an odd anti-hole with $K_1$. 8
Figure 3: The opposite of Lemma 2 is not true. In other words, we can not omit the subgraph condition of Theorem 8.

**Lemma 2.** Let $G$ be a neighborhood perfect graph. Then for all $k \in \mathbb{N}_0$, $G$ is $F(\omega, k)$-free if and only if $G$ is $F(\chi, k)$-free.

**Proof.** Let $G$ be a neighborhood perfect graph and let $H$ be an induced subgraph of $G$ that contains a dominating vertex, say $v$. Note that $H$ is neighborhood perfect and hence, $H - v$ is perfect. Let $k \in \mathbb{N}_0$ be such that $H \in F(\omega, k)$ or $H \in F(\chi, k)$. In this case, by Theorem 3, $H \in F(\chi, k)$ or $H \in F(\omega, k)$, respectively.

Note that the opposite of Lemma 2 is not true. Consider for example the graph drawn in Figure 3, that is, the join of a $K_1$ with the union of a $K_4$ and a $C_5$. This graph, say $G$, is not neighborhood perfect, since $W_5$ is an induced subgraph of $G$. But, since $\Delta(G) = 9$, $\omega(G) = 5$, $G \in \Omega_k$ and $G \in \Upsilon_k$ for all $k \geq 5$, and $G \not\in \Omega_k$ and $G \not\in \Upsilon_k$ for $k \leq 4$. That is, $G \not\in \Omega_k$ if and only if $G \not\in \Upsilon_k$, for all $k \in \mathbb{N}_0$. In other words, $G$ is not neighborhood perfect, but for all $k \in \mathbb{N}_0$, $G$ is $F(\omega, k)$-free if and only if $G$ is $F(\chi, k)$-free.

However, if we expand this property to all induced subgraphs of $G$, we find a characterization of neighborhood perfect graphs, as shown in the next theorem. Its proof needs a short preparation. The complement of a graph $G$ is denoted by $\overline{G}$. Consider the graph $W_{2r+3}$, $r \geq 1$, that is, the wheel with $2r + 4$ vertices, and the graph $B_{2r+3} = K_1 \bigoplus C_{2r+3}$, $r \geq 1$, that is, $B_{2r+3}$ is the join of one vertex with all vertices of an anti-hole with $2r + 3$ vertices. These graphs obey the three conditions of Theorem 2, if $k = 2r$ or $k = r + 1$, respectively. Thus $W_{2r+3} \in F(\omega, 2r)$ and $B_{2r+3} \in F(\omega, r+1)$. Since $\chi(W_{2r+3}) = 4, W_{2r+3} \not\in F(\chi, 2r)$. If we draw the vertices of $B_{2r+3} - v$ in a cycle and order the vertices such that every vertex is in between the two vertices not adjacent to it, we find an optimal coloring with $r + 1$ colors the following way. Start with one arbitrary vertex, name it with 1, go to its clockwise neighbor, name this one 2, and so on, until you reach number $2r + 3$, what is the neighbor of 1 again. Assign color 1 to vertices 1 and 2, color 2 to vertices 3 and 4 and so on. Observe that vertex $2r + 3$ is the only vertex to receive color $r + 1$, and $v$ is colored with $r + 2$, and that this coloring is an optimal coloring of $B_{2r+3}$. Hence, $B_{2r+3} \not\in F(\chi, r+1)$.

**Observation 3.** Let $k \in \mathbb{N}$, $k \geq 2$. Then $W_{k+3} \in F(\omega, k) \setminus F(\chi, k)$ and
We are now in the position to state Theorem 8.

**Theorem 8.** Let $G$ be a graph. Then the following statements are equivalent:

1. $G$ is a perfect neighborhood graph.
2. For all $k \in \mathbb{N}_0$ and all induced subgraphs $H$ of $G$, $H \in \Omega_k$ if and only if $H \in \Upsilon_k$.

**Proof.** Let $G$ be a perfect neighborhood graph and let $H$ be an induced subgraph of $G$. Hence, $H$ is also a perfect neighborhood graph. By Lemma 2, for every $k \in \mathbb{N}_0$, $H$ is $F(\omega, k)$-free if and only if $H$ is $F(\chi, k)$-free.

Let on the other hand $G$ be a graph that is not neighborhood perfect. Then, for some $k \geq 1$, $G$ contains a subgraph, say $H$, such that $H \cong W_{k+3}$ or $H \cong B_{2k+1}$ and hence a graph that is contained in $F(\omega, k)$, but not in $F(\chi, k)$. Hence, $H \in \Upsilon_k \setminus \Omega_k$. This completes the proof.

4. Final remarks

We introduced a sequence of new graph families. A member of such a family has the property that for some fixed $k \in \mathbb{N}_0$, the graph and all its induced subgraphs comply with $\Delta \leq \chi + k - 1$. We showed that those graphs can be characterized by a finite set of minimal forbidden induced subgraphs.

Moreover, we found some relations to the results presented in [6], where instead of $\Delta \leq \chi + k - 1$, we require $\Delta \leq \omega + k - 1$ for every induced subgraph. In particular, the neighborhood perfect graphs are exactly those graphs for which $\Upsilon_k$ and $\Omega_k$ coincide for all $k \in \mathbb{N}_0$ and all induced subgraphs of the graph.

A future direction might restrict the graph classes $\Upsilon_k$ to some graph universe like the claw-free graphs. There, a minimal forbidden induced subgraph $G$ has a unique dominating vertex $v$, $\Delta(G) = \chi(G) + k$ but in every optimal coloring of $G - v$, every color class contains exactly two vertices. This might lead to interesting results concerning the structure of minimal forbidden induced subgraphs.

Also, the question arises if some further results can be found if we compare $\Upsilon_k$ to $\Omega_j$ for some $j \in \mathbb{N}_0$, where $j \neq k$.

Finally, it might be of interest to focus on further graph parameters. We currently try to adapt our methods to the complement graph parameters of $\omega$ and $\chi$, that is, replacing $\omega$ or $\chi$ by the maximum size of an independent set or the clique cover number of the graph. In particular, we try to generalize our results, focusing on monotone graph parameters, where in our understanding, a parameter $\phi$ is monotone if for every induced subgraph $H$ of some graph $G$, $\phi(H) \leq \phi(G)$.

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