Triangle-free graphs that do not contain an induced subdivision of K_4 are 3-colorable

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Abstract

We show that triangle-free graphs that do not contain an induced subgraph isomorphic to a subdivision of K_4 are 3-colorable. This proves a conjecture of Trotignon and Vušković [5].

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1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. For a vertex $v \in V(G)$, we denote its set of neighbors by $N(v)$, and we let $N[v] = \{v\} \cup N(v)$. For $X, Y \subseteq V(G)$, we say that X is *complete* to Y if every vertex in X is adjacent to every vertex in Y; X is *anticomplete* to Y if every vertex in X is non-adjacent to every vertex in Y. A vertex $v \in V(G)$ is *complete* (*anticomplete*) to $X \subseteq V(G)$ if $\{v\}$ is complete (anticomplete) to X. A set $X \subseteq V(G)$ is a *cutset* for G if there is a partition (X, Y, Z) of $V(G)$ with $Y, Z \neq \emptyset$ and Y anticomplete to Z. The cutset X is a *clique cutset* if X is a (possibly empty) clique in G. For a graph H, we say that G *contains* H if H is isomorphic to an induced subgraph of G, and otherwise, G is *H-free*. For a family $\mathcal{F}$ of graphs, we say that G is *$\mathcal{F}$-free* if G is F-free for every graph $F \in \mathcal{F}$.

For a graph G and $X \subseteq V(G)$, $G|X$ denotes the induced subgraph of G with vertex set X. For $X \subseteq V(G)$, we let $G \setminus X = G|(V(G) \setminus X)$ and for $x \in V(G)$, we let $G \setminus x = G|(V(G) \setminus \{x\})$. By a *path* in a graph we mean an induced path. Let C be a cycle in G. The *length* of C is $|V(C)|$. The *girth* of G is the length of a shortest cycle, and is defined to be ∞ if G has no cycle. A *hole* in a graph is an induced cycle of length at least four. An ISK_4 is a graph that is isomorphic to a subdivision of K_4.

In [5] two of us studied the structure of ISK_4-free graphs, and proposed the following conjecture (and proved several special cases of it):

Conjecture 1. *If G is $\{\text{ISK}_4, \text{triangle}\}$-free, then $\chi(G) \leq 3$.*

In [2], Conjecture 1 was proved with 3 replaced by 4.

The main result of the present paper is the proof of Conjecture 1. In fact, we prove a stronger statement, from which Conjecture 1 easily follows:

Theorem 2. *Let G be an $\{\text{ISK}_4, \text{triangle}\}$-free graph. Then either G has a clique cutset, G is complete bipartite, or G has a vertex of degree at most two.*

For an induced subgraph H of G we write $v \in H$ to mean $v \in V(H)$. We use the same convention if H is a path or a hole. For a path $P = p_1 - \ldots - p_k$ we call the set $V(P) \setminus \{p_1, p_k\}$ the *interior* of P, and denote it by P^*.

A *wheel* in a graph is a pair $W = (C, x)$ where C is a hole and x has at least three neighbors in $V(C)$. We call C the *rim* of the wheel, and x the *center*. The neighbors of x in $V(C)$ are called the *spokes* of W. Maximal paths of C that do not contain any spokes in their interior are called the *sectors* of W. We write $V(W)$ to mean $V(C) \cup \{x\}$.

A graph is *series-parallel* if it does not contain a subdivision of K_4 as a (not necessarily induced) subgraph.

Theorem 3 ([1]). *Let G be a series-parallel graph. Then G is* ISK_4*-free, wheel-free, and $K_{3,3}$-free, and G contains a vertex of degree at most two.*

The following two facts were proved in [3]:

Theorem 4 ([3]). *Let G be an* $\{\text{ISK}_4, \text{triangle}\}$*-free graph. Then either G is series-parallel, or G contains a $K_{3,3}$ subgraph, or G contains a wheel. If G contains a subdivision of $K_{3,3}$ as an induced subgraph, then G contains a $K_{3,3}$.*

Theorem 5 ([3]). *If G is an* $\{\text{ISK}_4, \text{triangle}\}$*-free graph and G contains $K_{3,3}$, then either G is complete bipartite, or G has a clique cutset.*

Thus to prove Theorem 2 we need to analyze $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free graphs that contain wheels. This approach was already explored in [5], but we were able to push it further, as follows.

A wheel $W = (C, x)$ is *proper* if for every $v \in V(G) \setminus V(W)$

- there is a sector S of W such that $N(v) \cap V(C) \subseteq V(S)$.
- If v has at least three neighbors in $V(C)$, then v is adjacent to x.

(Please note that this definition is different from the one in [5].) We prove:

Theorem 6. *Let G be an* $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$*-free graph, and let x be the center of a proper wheel in G. If $W = (C, x)$ is a proper wheel with a minimum number of spokes subject to having center x, then*

1. *every component of $V(G) \setminus N(x)$ contains the interior of at most one sector of W, and*
2. *for every $u \in N(x)$, the component D of $V(G) \setminus (N(x) \setminus \{u\})$ such that $u \in V(D)$ contains the interiors of at most two sectors of W, and if S_1, S_2 are sectors with $S_i^* \subseteq V(D)$ for $i = 1, 2$, then $V(S_1) \cap V(S_2) \neq \emptyset$.*

Using Theorem 6 we can prove a variant of a conjecture from [5] that we now explain. For a graph G and $x, y \in V(G)$, we say that (x, y) is a *non-center pair* for G if neither x nor y is the center of a proper wheel in G, and $x = y$ or $xy \in E(G)$. We prove:

Theorem 7. *Let G be an* $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$*-free graph which is not series-parallel, and let (x, y) be a non-center pair for G. Then some $v \in V(G) \setminus (N[x] \cup N[y])$ has degree at most two.*

Here is the outline of the proof; the full proof is given in Section 5. We assume that G is a counterexample to Theorem 7 with $|V(G)|$ minimum. Since G is not series-parallel, it follows from Theorem 4 that G contains a wheel, and we show in Lemma 9 that G contains a proper wheel. Let $s \in V(G)$ be the center of a proper wheel chosen as in Theorem 6, and let $C_1, \dots, C_k$ be the components of $G \setminus N[s]$. By Theorem 6, it follows that $k > 1$. For each i, let N_i be the set of vertices of $N(s)$ with a neighbor in $V(C_i)$, and let $G_i = G|(V(C_i) \cup N_i \cup \{s\})$. We analyze the structure of the graphs G_i using the minimality of $|V(G)|$. It turns out that at most one G_i is not series-parallel, and that (by contracting C_i's) there is at most one value of i for which $|V(C_i)| > 1$. Also, if $|V(C_i)| > 1$, then $\{x, y\} \cap V(C_i) \neq \emptyset$. We may assume that $|V(C_i)| = 1$ for all $i \in \{1, \dots, k-1\}$, and that $\{x, y\} \cap V(C_k) \neq \emptyset$. Now consider the bipartite graph G', which (roughly speaking) is the graph obtained from $G \setminus \{s\}$ by contracting $V(C_k) \cup N_k$ to a single vertex z if $|V(C_k)| > 1$. It turns out that G' is $\{\text{ISK}_4, K_{3,3}\}$-free and has girth at least 6, while cycles that do not contain z must be even longer. Now either there is an easy win, or we find a cycle in G' that contains a long path P of vertices all of degree two in G' and with $V(P) \subseteq V(G) \setminus (N[x] \cup N[y])$. Further analysis shows that at least one of these vertices has degree two in G, and Theorem 7 follows.

This paper is organized as follows. In Section 2 we prove Theorem 6. Section 3 contains technical tools that we need to deduce that the graph G' described above has various useful properties. In Section 4 we develop techniques to produce a cycle with a long path of vertices of degree two. In Section 5 we put all of our knowledge together to prove Theorems 2 and 7 and deduce Conjecture 1.

Let us finish this section with an easy fact about ISK_4-free graphs. Given a hole C and a vertex $v \notin C$, v is *linked* to C if there are three paths P_1, P_2, P_3 such that

- $P_1^* \cup P_2^* \cup P_3^* \cup \{v\}$ is disjoint from C;
- each P_i has one end v and the other end in C, and there are no other edges between P_i and C;
- for $i, j \in \{1, 2, 3\}$ with $i \neq j$, $V(P_i) \cap V(P_j) = \{v\}$;
- if $x \in P_i$ is adjacent to $y \in P_j$ then either $v \in \{x, y\}$ or $\{x, y\} \subseteq V(C)$; and
- if v has a neighbor $c \in C$, then $c \in P_i$ for some i.

Lemma 8. *If G is* ISK_4*-free, then no vertex of G can be linked to a hole.*

2 Wheels

Lemma 9. *Let G be an $\{\text{ISK}_4, \text{triangle}\}$-free graph that contains a wheel. Then there is a proper wheel in G.*

Proof. Let G be an $\{\text{ISK}_4, \text{triangle}\}$-free graph. Let $W = (C, x)$ be a wheel in G with $|V(C)|$ minimum. We claim that W is a proper wheel. Suppose $v \in V(G) \setminus V(W)$ violates the definition of a proper wheel.

If v has at least three neighbors in the hole $x - S - x$ for some sector S of W, then $(x - S - x, v)$ is a wheel with shorter rim than W, a contradiction. So v has at most two neighbors in every sector of W (and at most one if v is adjacent to x). Therefore there exist sectors S_1, S_2 of W such that v has a neighbor in $V(S_1) \setminus V(S_2)$ and a neighbor in $V(S_2) \setminus V(S_1)$. Also by the minimality of $|V(C)|$, every path of C whose ends are in $N(v)$ and with interior disjoint from $N(v)$ contains at most two spokes of W, and we can choose S_1, S_2 and for $i = 1, 2$, label the ends of S_i as a_i, b_i such that either $b_1 = a_2$, or b_1, a_2 are the ends of a third sector S_3 of W and v has no neighbor in S_3^*. If possible, we choose S_1, S_2 such that $b_1 = a_2$. If v has two neighbors in S_1, denote them s, t such that a_1, t, s, b_1 are in order in C. If v has a unique neighbor in S_1, denote it by s. Let z be the neighbor of v in S_2 closest to a_2.

Assume first that v is non-adjacent to x. Suppose $b_1 \neq a_2$. By Lemma 8, x cannot be linked to the hole $z - S_2 - a_2 - S_3 - b_1 - S_1 - s - v - z$, and it follows that $z \neq b_2$. If v has two neighbors in S_1, then v can be linked to $x - S_3 - x$ via the paths $v - s - S_1 - b_1$, $v - t - S_1 - a_1 - x$, $v - z - S_2 - a_2$; and if v has a unique neighbor in S_1, then s can be linked to $x - S_3 - x$ via the paths $s - S_1 - b_1$, $s - S_1 - a_1 - x$, $s - v - z - S_2 - a_2$ (note that by the choice of S_1, S_2 and since $b_1 \neq a_2$, it follows that $s \neq b_1$). In both cases, this is contrary to Lemma 8. This proves that $b_1 = a_2$. Let y be the neighbor of v in S_2 closest to b_2. Now if v has two neighbors in S_1, then v can be linked to $x - S_1 - x$ via the paths $v - s$, $v - t$, $v - y - S_2 - b_2 - x$, contrary to Lemma 8. So v has a unique neighbor in S_1, and similarly a unique neighbor in S_2. It follows that s, b_1 and z are all distinct. Now we can link x to $s - S_1 - b_1 - S_2 - z - v - s$ via the paths $x - b_1$, $x - a_1 - S_1 - s$, and $x - b_2 - S_2 - z$, contrary to Lemma 8.

This proves that v is adjacent to x, and so v has at most one neighbor in every sector of W. If $b_1 \neq a_2$, then v can be linked to $x - S_3 - x$ via the paths $v - s - S_1 - b_1$, $v - x$, $v - z - S_2 - a_2$, and if $b_1 = a_2$, then s, b_1 and z are all distinct and hence x can be linked to the hole $s - S_1 - b_1 - S_2 - z - v - s$ via the paths $x - b_1$, $x - v$, and $x - b_2 - S_2 - z$; in both cases contrary to

Lemma 8. This proves that every $v \in V(G) \setminus V(W)$ satisfies the condition in the definition, and so W is a proper wheel in G. □

Let $W = (C, v)$ be a wheel. We call x *proper* for W if either $x \in V(C) \cup \{v\}$; or

- all neighbors of x in $V(C)$ are in one sector of W; and
- if x has more than two neighbors in $V(C)$, then x is adjacent to v.

A vertex x is *non-offensive* for a wheel $W = (C, v)$ if there exist two sectors S_1, S_2 of W such that

- x is adjacent to v;
- x has neighbors in S_1 and in S_2;
- $N(x) \cap V(C) \subseteq V(S_1) \cup V(S_2)$;
- S_1 and S_2 are consecutive; and
- if $u \in V(G) \setminus V(W)$ is adjacent to x, then $N(u) \cap V(C) \subseteq V(S_1) \cup V(S_2)$.

If $V(S_1) \cap V(S_2) = \{a\}$, we also say that x is *a-non-offensive*.

Lemma 10. *Let G be an $\{\text{ISK}_4, \text{triangle}\}$-free graph. Let $W = (C, v)$ be a wheel in G. Let S_1, S_2 be consecutive sectors of W, and let $x \in N(v) \setminus V(W)$ be a vertex such that $N(x) \cap V(C) \subseteq V(S_1) \cup V(S_2)$ and $N(x) \cap V(S_1), N(x) \cap V(S_2) \neq \emptyset$. Then $N(x) \cap V(C) \subseteq S_1^* \cup S_2^*$ and for $\{i, j\} = \{1, 2\}$, $|N(x) \cap (V(S_i) \setminus V(S_j))| \geq 3$.*

Proof. Since x is adjacent to v and G is triangle-free, it follows that $N(x) \cap V(C) \subseteq S_1^* \cup S_2^*$. Suppose for a contradiction that $|N(x) \cap (V(S_1) \setminus V(S_2))| \leq 2$.If $|N(x) \cap (V(S_1) \setminus V(S_2))| = 2$, then x has exactly three neighbors in the hole $v - S_1 - v$, contrary to Lemma 8. It follows that $|N(x) \cap (V(S_1) \setminus V(S_2))| = 1$. Let z denote the neighbor of x in $V(S_1)$. Let $\{w\} = V(S_1) \cap V(S_2)$, and let y denote the neighbor of x in $V(S_2)$ closest to w along S_2. Then x can be linked to the hole $v - S_1 - v$ via the three paths $x - v$, $x - z$, and $x - y - S_2 - w$. This is a contradiction to Lemma 8, and the result follows. □

We say that wheel $W = (C, v)$ is *k-almost proper* if there are spokes $x_1, \ldots, x_k$ of W and a set $X \subseteq V(G) \setminus V(W)$ such that

- no two spokes in $\{x_1, \ldots, x_k\}$ are consecutive;

- W is proper in $G \setminus X$;
- for every x in X there exists i such that x is x_i-non-offensive.

Lemma 11. *Let G be an $\{\text{ISK}_4, \text{triangle}\}$-free graph, and let $W = (C, v)$ be a 1-almost proper wheel in G. Let x_1 and X be as in the definition of a 1-almost proper wheel, and let S_1 and S_2 be the sectors of W containing x_1.*

Then there exists a proper wheel W' in G with center v and the same number of spokes as W. Moreover, either $W = W'$, or $V(W') \setminus V(W) = \{x^*\}$ where x^* is a non-offensive vertex for W, and $V(W) \setminus V(W') \subseteq V(S_1^*) \cup V(S_2^*) \cup \{x_1\}$.

Proof. We may assume that $X \neq \emptyset$, for otherwise W is proper in G. For $x \in X$, let $P(x)$ denote the longest path in $G|(V(S_1) \cup V(S_2))$ starting and ending in a neighbor of x. Let $x^* \in X$ be a vertex with $|V(P(x^*))|$ maximum among vertices in X, and let Y denote interior of $P(x^*)$. Let $C' = G|((V(C) \cup \{x^*\}) \setminus Y)$. It follows that $W' = (C', v)$ is a wheel. Moreover, $N(v) \cap V(C') = ((N(v) \cap V(C)) \setminus \{x_1\}) \cup \{x^*\}$, and therefore W' has the same number of spokes as W.

If W' is proper, the result follows. Therefore, we may assume that there is a vertex $y \in V(G) \setminus V(W')$ that is not proper for W'. Let S_1', S_2' denote the sectors of W' containing $S_1 \setminus Y$ and $S_2 \setminus Y$, respectively.

Suppose first that $y \in V(W)$, and consequently $y \in Y$. Since x^* has at least two neighbors in each of S_1 and S_2 by Lemma 10, it follows that $|V(P(x^*))| \geq 4$. Consequently, either $N(y) \cap V(C') \subseteq S_1'$ or $N(y) \cap V(C') \subseteq S_2'$. Moreover, $N(y) \cap V(C') \subseteq N[x]$, and therefore $|N(y) \cap V(C')| \leq 1$. This implies that y is proper for W, a contradiction. This proves that $y \notin V(W)$.

Next, we suppose that $y \in X$. It follows that $N(y) \cap V(C') \subseteq V(S_1') \cup V(S_2')$. Since y is not proper for W', but y is adjacent to v, it follows that $N(y) \cap V(S_1'), N(y) \cap V(S_2') \neq \emptyset$. By Lemma 10, y has at least two neighbors in S_1'. But then $V(P(x^*)) \subsetneq V(P(y))$, a contradiction to the choice of x^*. This proves that $y \in V(G) \setminus (X \cup V(W))$.

If $y \notin N(x^*)$, then $N(y) \cap V(C') \subseteq N(y) \cap V(C)$, and since $y \notin X$, it follows that y is proper for W and thus y is proper for W'. Consequently $y \in N(x^*)$. Since x^* is non-offensive for W, it follows that $N(y) \cap V(C) \subseteq V(S_1) \cup V(S_2)$. Since y is proper for W, we may assume by symmetry that $N(y) \cap V(C) \subseteq V(S_1)$. It follows that $N(y) \cap V(C') \subseteq V(S_1')$. Since y is adjacent to x^* and G is triangle-free, it follows that y is non-adjacent to v, and hence $|N(y) \cap V(C)| \leq 2$. This implies that $|N(y) \cap V(C')| \leq 3$, and by Lemma 8, it is impossible for y to have exactly three neighbors in C' since G is ISK_4-free. Therefore, $|N(y) \cap V(C')| \leq 2$, and therefore y is proper for W'. This is a contradiction, and it follows that W' is proper in G, and hence W' is the desired wheel. $\square$

Lemma 12. *Let G be an $\{\mathrm{ISK}_4, \text{triangle}\}$-free graph, and let $W = (C, v)$ be a 2-almost proper wheel in G. Then there exists a proper wheel in G with center v and at most the same number of spokes as W.*

Proof. Let x_1, x_2 and X be as in the definition of a 2-almost proper wheel, and let S_1, S_2 be the sectors of W containing x_1. Let X_1 denote the set x_1-non-offensive vertices in X, and let $X_2 = X \setminus X_1$. We may assume that X_1, X_2 are both non-empty, for otherwise the result follows from Lemma 11. It follows that W is 1-almost proper, but not proper, in $G \setminus X_2$. Let W', x^* be as in Lemma 11. So W' is a proper wheel in $G \setminus X_2$. If W' is 1-almost proper in G, then the result of the Lemma follows from Lemma 11. So we may assume that W' is not 1-almost proper in G. Since every vertex of $V(G) \setminus X_2$ is proper for W', we deduce that some vertex of $x \in X_2$ is not proper and not x_2-non-offensive for W'.

By the definition of X_1, and since W is 2-almost proper in G, it follows that $N(x) \cap V(C)$ is contained in the sectors S_3, S_4 of W containing x_2. Since x_1 and x_2 are not consecutive, $S_3, S_4 \notin \{S_1, S_2\}$, and so by Lemma 11, $V(S_3) \cup V(S_4) \subseteq V(W')$. Consequently, S_3 and S_4 are sectors of W'. Since G is triangle-free and every vertex in X is adjacent to v, it follows that x is not adjacent to x^*. Therefore, $N(x) \cap V(C') \subseteq V(S_3) \cup V(S_4)$, x has both a neighbor in S_3 and a neighbor in S_4, and S_3, S_4 are the sectors of W' containing x_2. Let s_3 denote the neighbor of x in S_3 furthest from x_2, and let s_4 denote the neighbors of x in S_4 furthest from x_2. We may assume that among all vertices of X_2 that are not x_2-non-offensive for W', x is chosen so that the path of C from s_3 to s_4 containing x_2 is maximal.

Since x is not x_2-non-offensive for W', there exists a vertex $u \in N(x) \setminus V(W')$ with a neighbor in $V(C') \setminus (V(S_3) \cup V(S_4))$. Since x is x_2-non-offensive for W, it follows that u has a neighbor in $V(C') \setminus V(C) = \{x^*\}$, and so u is adjacent to x and x^*.

Since x and x^* are non-offensive for W, it follows that $N(u) \cap V(C) \subseteq (V(S_1) \cup V(S_2)) \cap (V(S_3) \cup V(S_4))$. Since G is triangle-free, u is non-adjacent to v, and therefore $u \notin X$. Consequently, u is proper for W, and all the neighbors of u in C belong to one sector of W. It follows that u has at most one neighbor in $V(C)$. Suppose that u has exactly one neighbor in $V(C)$. Then u has three neighbors in the cycle arising from C' by replacing $s_3 - S_3 - x_2 - S_4 - s_4$ by $s_3 - x - s_4$, contrary to Lemma 8. It follows that u has no neighbors in $V(C)$.

Let P'_1 denote the path of C' from s_3 to x^* not containing x_2, and let P_1 be $x - s_3 - P'_1 - x^*$. Let P'_2 denote the path of C' from s_4 to x^* not containing x_2, and let P_2 be $x - s_4 - P'_1 - x^*$. Let $D = G|(V(P_1) \cup \{u\})$. Since x_1 and x_2 are not consecutive, each of P_1^*, P_2^* contains at least one neighbor of v, and so $W'' = (D, v)$ is a wheel with fewer spokes than W. Let S'_3 denote the sector of W'' containing x but not containing u. If W'' is proper in G, then the result follows. Therefore, we may assume that there is a vertex $y \in V(G) \setminus V(W'')$ that is not proper for W''.

Since every vertex in $V(W') \setminus V(W'')$ and every vertex in $V(W) \setminus V(W'')$ has at most one neighbor in $V(W'')$, it follows that $y \notin V(W) \cup V(W')$. Suppose that $y \notin N(u)$. If $y \in X_2$, then $N(y) \cap V(D) \subseteq (V(S_3) \cup \{x\}) \cap V(D)$, and so y is proper for W'', since y is adjacent to v, a contradiction. Thus $y \notin X_2$, and so y is proper for W'. If $y \notin N(x)$, then $N(y) \cap V(D) \subseteq N(y) \cap V(C')$, and again y is proper for W'', a contradiction. Thus $y \in N(x)$, but since x is non-offensive for W, $N(y) \cap V(C) \subseteq V(S_3) \cup V(S_4)$, and so $N(y) \cap V(D) \subseteq V(S'_3)$. Since y is not proper for W'', y is non-adjacent to v and has at least three neighbors in S'_3. But y is proper for W', and so y has at most two neighbors in S_3; thus y has exactly three neighbors in S'_3 and hence in D contrary to Lemma 8. This contradiction implies that $y \in N(u)$.

Since y is not proper for W'', it follows that y has a neighbor in P_1^*, and since G is triangle-free, it follows that y is non-adjacent to x, x^*. We claim that y has no neighbor in P_2. Suppose that it does. If $y \in X_2$, then, since y is adjacent to u and has a neighbor in P_1^*, we deduce that y is not x_2-non-offensive for W', and the claim follows from the maximality of the path of C from s_3 to s_4 containing x_2. Thus we may assume that $y \notin X_2$. Consequently, y is proper for W', a contradiction. This proves the claim.

Let z_1 be the neighbor of y in $V(P_1)$ closest to x along P_1, and let z_2 be the neighbor of y in $V(P_1)$ closest to x^* along P_1. Let D' be the hole $x^* - P_2 - x - u - x^*$. If $z_1 \neq z_2$, we can link y to D' via the paths $y - z_1 - P_1 - x$, $y - z_2 - P_1 - x^*$ and $y - u$, and if $z_1 = z_2$, then we can link z_1 to D' via the paths $z_1 - P_1 - x$, $z_1 - P_1 - x^*$ and and $z_1 - y - u$, in both cases contrary to Lemma 8. This proves Lemma 12. □

Throughout the remainder of this section G is an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free graph, and $W = (C, x)$ is a proper wheel in G with minimum number of spokes (subject to having center x).

Lemma 13. *Let $P = p_1 - \ldots - p_k$ be a path such that p_1, p_k have neighbors in $V(C)$, $V(P) \subseteq V(G) \setminus V(W)$, and there are no edges between P^* and $V(C)$. Assume that no sector of W contains $(N(p_1) \cup N(p_k)) \cap V(C)$. For $i \in \{1, k\}$, if x is non-adjacent to p_i, then p_i has a unique neighbor in C.*

Proof. Let S_1, S_2 be distinct sectors of W such that $N(p_1) \cap V(C) \subseteq V(S_1)$, and $N(p_k) \cap V(C) \subseteq V(S_2)$. We may assume that p_1 is non-adjacent to x, and so p_1 has at most two neighbors in C. Since p_1 cannot be linked to the hole C (or the hole obtained from C by rerouting S_2 through p_k) via two one-edge paths and P, it follows that p_1 has a unique neighbor in C. □

Theorem 14. *Let $P = p_1 - \ldots - p_k$ be a path with $V(P) \subseteq V(G) \setminus V(W)$ such that x has at most one neighbor in P.*

1. *If P contains no neighbor of x, then there is a sector S of W such that every edge from P to C has an end in $V(S)$.*
2. *If P contains exactly one neighbor of x, then there are two sectors S_1, S_2 of W such that $V(S_1) \cap V(S_2) \neq \emptyset$, and every edge from P to C has an end in $V(S_1) \cup V(S_2)$ (where possibly $S_1 = S_2$).*

Proof. Let P be a path violating the assertions of the theorem and assume that P is chosen with k minimum. Since W is proper, it follows that $k > 1$. Our first goal is to show that x has a neighbor in $V(P)$.

Suppose that x is anticomplete to $V(P)$. Then, by the minimality of k, there exist two sectors S_1, S_2 of W such that every edge from $\{p_1, \ldots, p_{k-1}\}$ to $V(C)$ has an end in $V(S_1)$, and every edge from $\{p_2, \ldots, p_k\}$ to $V(C)$ has an end in $V(S_2)$. It follows that $S_1 \neq S_2$. Then p_1 has a neighbor in $V(S_1) \setminus V(S_2)$, and p_k has a neighbor in $V(S_2) \setminus V(S_1)$, and every edge from P^* to $V(C)$ has an end in $V(S_1) \cap V(S_2)$. For $i = 1, 2$ let a_i, b_i be the ends of S_i. We may assume that a_1, b_1, a_2, b_2 appear in C in this order and that $a_1 \neq b_2$. Let Q_1 be the path of C from b_2 to a_1 not using b_1, and let Q_2 be the path of C from b_1 to a_2 not using a_1. We can choose S_1, S_2 with $|V(Q_2)|$ minimum (without changing P). Let s be the neighbor of p_1 in S_1 closest to a_1, t the neighbor of p_1 in S_1 closest to b_1, y the neighbor of p_k in S_2 closest to a_2 and z the neighbor of p_k in S_2 closest to b_2. Then $s \neq b_1$ and $z \neq a_2$. It follows that $V(Q_2) \cap \{s, z\} = \emptyset$. Moreover, if $V(S_1) \cap V(S_2) \neq \emptyset$, then $b_1 = a_2$ and $V(Q_2) = \{b_1\}$, and in all cases $V(Q_1)$ is anticomplete to P^*.

Now $D_1 = s - p_1 - P - p_k - z - S_2 - b_2 - Q_1 - a_1 - S_1 - s$ is a hole.

(1) *$W_1 = (D_1, x)$ is a wheel with fewer spokes than W.*

Since $V(Q_2) \cap V(D_1) = \emptyset$ and $V(Q_2)$ contains a neighbor of x, it follows that x has fewer neighbors in D_1 than it does in C. It now suffices to show that x has at least three neighbors in Q_1. Since $a_1, b_2 \in V(Q_1)$, we may assume that x has no neighbor in Q_1^*, and Q_1 is a sector of W. Since not every edge between $V(P)$ and $V(C)$ has an end in $V(Q_1)$, it follows that

$t \neq a_1$ or $y \neq b_2$. By symmetry, we may assume that $t \neq a_1$. Since x cannot be linked to W by Lemma 8, it follows that x has at least four neighbors in $V(C)$, and therefore $V(S_1) \cap V(S_2) = \emptyset$. Consequently, P^* is anticomplete to $V(C)$. It follows from Lemma 13 that $s = t$. Now we can link s to the hole $a_1 - Q_1 - b_2 - x - a_1$ via the paths $s - S_1 - a_1$, $s - p_1 - P - p_k - z - S_2 - b_2$ and $s - S_1 - b_1 - x$, contrary to Lemma 8. This proves that $W_1 = (D_1, x)$ is a wheel with fewer spokes than W. This proves (1).

By the choice of W, it follows from (1) that W_1 is not proper. Let S_0 be the sector $a_1 - S_1 - s - p_1 - P - p_k - z - S_2 - b_2$ of (D_1, x). Since W is a proper wheel and W_1 is not a proper wheel, we have that:

(2) *There exists $v \in V(G) \setminus V(W_1)$ such that either*

- *v is non-adjacent to x, and v has at least three neighbors in S_0 and $N(v) \cap V(D_1) \subseteq V(S_0)$, or*
- *there is a sector S_3 of W with $V(S_3) \subseteq V(Q_1)$, such that v has a neighbor in $V(S_3) \setminus V(S_0)$ and a neighbor in $V(S_0) \setminus V(S_3)$, and $N(v) \cap V(C) \subseteq V(S_3)$.*

Let v be as in (2). First we show that $v \notin V(C)$. The only vertices of C that may have more than one neighbor in D_1 are b_1 and a_2, and that only happens if $b_1 = a_2$. But $N(b_1) \cap V(D_1) \subseteq V(S_0)$ and b_1 is adjacent to x, so b_1 does not satisfy the conditions described in the bullets. Thus $v \notin V(C)$.

(3) *v has a unique neighbor in P.*

If the first case of (2) holds, then the statement of (3) follows immediately from the minimality of k (since v is non-adjacent to x), and so we may assume that the second case of (2) holds. Observe that no vertex of $V(Q_1)$ is contained both in a sector with end a_1 and in a sector with end b_2, and therefore we may assume that v has a neighbor in a sector that does not have end b_2. If v is non-adjacent to x, we get a contradiction to the minimality of k. So we may assume that v is adjacent to x, and therefore v has a neighbor in S_3^*, and $b_2 \notin V(S_3)$. Let S_4 be the sector of D_1 such that $b_2 \in V(S_4)$ and $V(S_4) \subseteq V(Q_1)$. Suppose that $y \neq b_2$ or $V(S_3) \cap V(S_4) = \emptyset$. Let $i \in \{1, \dots, k\}$ be maximum such that v is adjacent to p_i. Now the path $v - p_i - P - p_k$ violates the assertions of the theorem, and so it follows from the minimality of k that $N(v) \cap V(P) \subseteq \{p_1, p_2\}$. Therefore, since G is triangle-free, it follows that v has a unique neighbor in P, and (3) holds. So we may assume that $y = b_2$ and there exists $a_3 \in V(C)$ such that $V(S_4) \cap V(S_3) = \{a_3\}$. Let R be the path from v to a_3 with $R^* \subseteq S_3^*$. Now we can link v to $x{-}S_4{-}x$ via the paths $v{-}x$, $v{-}R{-}a_3$ and $v{-}p_i{-}P{-}p_k{-}b_2$, where i is maximum such that v is adjacent to p_i, contrary to Lemma 8. This proves (3).

In view of (3) let $N(v) \cap V(P) = \{p_j\}$. In the case of the first bullet of (2), since v cannot be linked to the hole $x - S_0 - x$ by Lemma 8, it follows that v has at least four neighbors in S_0, and therefore at least three neighbors in $V(S_1) \cup V(S_2)$, contrary to the fact that W is proper. So the case of the second bullet of (2) holds. Since W is proper $N(v) \cap (V(S_0) \setminus V(S_3)) \subseteq V(P)$, and $N(v) \cap V(D_1) \subseteq V(S_0) \cup V(S_3)$.

(4) *There are edges between P^* and $V(C)$.*

Suppose not. By Lemma 13, $s = t$ and $y = z$. We claim that in this case $b_1 \neq a_2$, for if $b_1 = a_2$, then b_1 can be linked to the hole $x - a_1 - S_1 - s - p_1 - P - p_k - z - S_2 - b_2 - x$ via the paths $b_1 - x$, $b_1 - S_1 - s$ and $b_1 - S_2 - z$, contrary to Lemma 8. If v has a unique neighbor r in C, then p_j can be linked to C via the paths $p_j - P - p_1 - s$, $p_j - P - p_k - z$ and $p_j - v - r$, contrary to Lemma 8, so v has at least two neighbors in C. Recall that $N(v) \cap V(C) \subseteq V(S_3)$. Let D be the hole obtained from C by rerouting S_3 through v. Then $s, z \in V(D)$, and p_j can be linked to D via the paths $p_j - P - p_1 - s$, $p_j - P - p_k - z$ and $p_j - v$, contrary to Lemma 8. This proves (4).

If follows from (4) that $b_1 = a_2$ and b_1 has neighbors in P^*. Now, by considering the path from a neighbor of b_1 in P^* to v with interior in P^* if v has a neighbor in P^*, and the paths $v - p_1$ or $v - p_k$ if v has no neighbor in P^*, the minimality of k implies that v is adjacent to x and one of a_1, b_2 belongs to S_3.

By symmetry we may assume $a_1 \in V(S_3)$. Let R be the path from v to a_1 with $R^* \subseteq V(S_3)$. Now x can be linked to the hole $v - R - a_1 - S_1 - s - p_1 - P - p_j - v$ via the paths $x - v$, $x - a_1$ and $x - b_2 - S_2 - z - p_k - P - p_j$, contrary to Lemma 8.

In summary, we have now proved:

(5) *If P' is a path violating the assertion of the theorem and $|V(P')| = k$, then x has a neighbor in $V(P')$.*

By (5), x has a neighbor in $V(P)$, say x is adjacent to p_i. Then p_i is the unique neighbor of x in $V(P)$. By the minimality of k, there exist two distinct sectors S_1, S_2 of W such that p_1 has a neighbor in $V(S_1) \setminus V(S_2)$, and p_k has a neighbor in $V(S_2) \setminus V(S_1)$. By (5), if $1 < i < j$, then every edge from $\{p_1, \ldots, p_{i-1}\}$ to $V(C)$ has an end in $V(S_1)$, and every edge from $\{p_{i+1}, \ldots, p_k\}$ to $V(C)$ has an end in $V(S_2)$; if $i = 1$ then every edge from $V(P) \setminus \{p_1\}$ to $V(C)$ has and end in $V(S_2)$; and if $i = k$ then every edge from $V(P) \setminus \{p_k\}$ to $V(C)$ has and end in $V(S_1)$.

For $j = 1, 2$, let a_j, b_j be the ends of S_j.

(6) *One of the following statements holds:*

- *there are no edges between $V(C)$ and P^*, or*
- *we can choose S_1, S_2 such that a_1, b_1, a_2, b_2 appear in C in order and there is a sector S_3 with ends b_1, a_2, and every edge between $V(C)$ and P^* is from b_1 to $\{p_2, \ldots, p_{i-1}\}$ or from p_i to S_3^*, or from a_2 to $\{p_{i+1}, \ldots, p_{k-1}\}$.*

Suppose (6) is false. It follows that there are edges between P^* and $V(C)$. Since G is triangle-free, p_i is anticomplete to $N(x) \cap V(C)$. Suppose that there is sector S_3 of W and an edge from S_3^* to P^*. By the minimality of k we deduce that $S_3 \notin \{S_1, S_2\}$, $1 < i < k$ and p_i has a neighbor in S_3^*. Again by the minimality of k it follows that there exist sectors S_1', S_2' such that $V(S_j') \cap V(S_3) \neq \emptyset$ for $j = 1, 2$ and every edge from $\{p_1, \ldots, p_{i-1}\}$ to C has an end in S_1', and every edge from $\{p_{i+1}, \ldots, p_k\}$ to C has an end in S_2'. Now we can choose $S_1 = S_1'$ and $S_2 = S_2'$. We may assume that a_1, b_1, a_2, b_2 appear in C in this order, and so b_1 and a_2 are the ends of S_3. Since p_i has a neighbor in S_3^*, the minimality of k implies that $\{p_2, \ldots, p_i\}$ is anticomplete to $V(S_1) \setminus \{b_1\}$, and $\{p_i, \ldots, p_{k-1}\}$ is anticomplete to $V(S_2) \setminus \{a_2\}$, and the second bullet is satisfied. So P^* is anticomplete to $V(C) \setminus N(x)$. Since there are edges between P^* and $V(C)$, and since p_i is anticomplete to $N(x) \cap V(C)$, by symmetry we may assume that there is an edge between $\{p_2, \ldots, p_{i-1}\}$ and $t \in N(x) \cap V(C)$. Then $t \in V(S_1)$. Let S_3 be the other sector of W incident with t. By the minimality of k it follows that S_2 can be chosen so that $V(S_3) \cap V(S_2) \neq \emptyset$, and again the case of the second bullet holds. This proves (6).

If the second bullet of (6) holds, let Q_1 be the path of C from b_2 to a_1 not using b_1, and let $Q_2 = S_3$. To define Q_1 and Q_2, let us now assume that the case of the first bullet holds. We may assume that a_1, b_1, a_2, b_2 appear in C in this order. Also, a_1, b_1, a_2, b_2 are all distinct, since P violates the assertion of the theorem. Let Q_1 be the path of C from b_2 to a_1 not using

b_1, and let Q_2 be the path of C from b_1 to a_2 not using a_1. We may assume that S_1, S_2 are chosen with $|V(Q_2)|$ minimum (without changing P).

Since W is proper, it follows that $N(p_1) \cap V(C) \subseteq V(S_1)$ and $N(p_k) \cap V(C) \subseteq V(S_2)$. Let s be the neighbor of p_1 in S_1 closest to a_1, t the neighbor of p_1 in S_1 closest to b_1, y the neighbor of p_k in S_2 closest to a_2 and z the neighbor of p_k in S_2 closest to b_2. Then $s \neq b_1$ and $z \neq a_2$.

Let D_1 be the hole $a_1 - S_1 - s - p_1 - P - p_k - z - S_2 - b_2 - Q_1 - a_1$. Then $W_1 = (D_1, x)$ is a wheel with fewer spokes than W. We may assume that (subject to the minimality of k) P was chosen so that $V(Q_1)$ is (inclusion-wise) minimal. By Lemma 8, x has a neighbor in $V(D_1) \setminus \{a_1, b_1, p_i\}$, and so x has a neighbor in Q_1^*.

Let S_0 be the sector $a_1 - S_1 - s - p_1 - P - p_i$, and let T_0 be the sector $p_i - P - p_k - z - b_2$ of (D_1, x).

(7) *No vertex $v \in V(G) \setminus V(W_1)$ has both a neighbor in $V(S_0) \setminus V(T_0)$ and a neighbor in $V(T_0) \setminus V(S_0)$.*

Suppose (7) is false, and let $v \in V(G) \setminus V(W_1)$ be such that v has a neighbor in $V(S_0) \setminus V(T_0)$ and a neighbor in $V(T_0) \setminus V(S_0)$.

First we claim that v is adjacent to x. Suppose v has a neighbor in $V(a_1 - S_1 - s)$. Since W is proper and $a_1, s \notin V(S_2)$ (because P violates the statement of the theorem), it follows that v has no neighbor in $V(z - S_2 - b_2)$. Consequently v has a neighbor in $V(T_0) \setminus (V(S_2) \cup V(S_0))$. Let j be maximum such that v is adjacent to p_j, then $j > i$. Now applying (5) to the path $v - p_j - P - p_k$ we deduce that v is adjacent to x, as required. Thus we may assume that $N(v) \cap (V(S_0) \cup V(T_0)) \subseteq V(P)$. Let j be minimum and l maximum such that v is adjacent to p_j, p_l. Then $j < i$ and $l > i$. Applying (5) to the path $p_1 - P - p_j - v - p_l - P - p_k$, we again deduce that x is adjacent to v. This proves the claim.

In view of the claim, Lemma 10 implies that v has at least two neighbors in $V(T_0) \setminus V(S_0)$ and at least two neighbors in $V(S_0) \setminus V(T_0)$. But now, rerouting P through v (as in the previous paragraph), we get a contradiction to the minimality of k. This proves (7).

(8) *Every non-offensive vertex for W_1 is either a_1-non-offensive or b_2-non-offensive.*

Let v be a non-offensive vertex for W_1. Since W is proper, it follows that $N(v) \cap V(C)$ is included in a unique sector of W. Consequently, v

is either a_1-non-offensive, or b_2-non-offensive, or p_i non-offensive. However, (7) implies that v is not p_i-non-offensive, and (8) follows.

Let X be the set of all non-offensive vertices for W_1. It follows from Lemma 12 that W_1 is not proper in $V(G) \setminus X$.

(9) *There exists $v \in V(G) \setminus (V(W_1) \cup X)$ such that one of the following holds:*

- *v is non-adjacent to x, and v has at least three neighbors in S_0, and $N(v) \cap V(D_1) \subseteq V(S_0)$.*
- *v is non-adjacent to x, and v has at least three neighbors in T_0, and $N(v) \cap V(D_1) \subseteq V(T_0)$.*
- *v has a neighbor in $V(S_0) \setminus V(T_0)$ and a neighbor in $V(T_0) \setminus V(S_0)$, and $N(v) \cap V(D_1) \subseteq V(S_0) \cup V(T_0)$.*
- *(possibly with the roles of S_0 and T_0 exchanged) there is a sector S_4 of W with $V(S_4) \subseteq V(Q_1)$ such that v has a neighbor in $V(S_4) \setminus (V(S_0) \cup V(T_0))$, v has a neighbor in $V(S_0) \setminus V(S_4)$, v does not have a neighbor in $V(T_0) \setminus (V(S_0) \cup V(S_4))$, and $N(v) \cap V(C) \subseteq V(S_4)$.*

We may assume that the first three bullets of (9) do not hold. Since W is proper and W_1 is not, (possibly switching the roles of S_0 and T_0) there exists $v \in V(G) \setminus V(W_1)$ and a sector S_4 of W with $V(S_4) \subseteq V(Q_1)$, such that v has a neighbor in $V(S_4) \setminus V(S_0)$, v has a neighbor in $V(S_0) \setminus V(S_4)$, and $N(v) \cap V(C) \subseteq V(S_4)$. But now (7) implies that the last bullet of (9) holds. This proves (9).

Let $v \in V(G)$ be as in (9). Next we show that:

(10) *v has a unique neighbor in $V(P)$.*

Suppose that v has at least two neighbors in P. In the first two cases of (9) we get a contradiction to the minimality of k. The third case is impossible by (7). Thus we may assume that the case of the fourth bullet of (9) holds. We may assume that $N(v) \cap V(P) \subseteq V(S_0)$, and in particular v has a neighbor in $\{p_1, \dots, p_{i-1}\}$. Suppose first that v is non-adjacent to x. Since v has a neighbor in $V(S_4) \setminus V(S_0)$, the minimality of k implies that $t = a_1$ and $a_1 \in V(S_4)$, and also that $b_2 \in V(S_4)$, contrary to the fact that x

has a neighbor in Q_1^*. So v is adjacent to x, and therefore v has a neighbor in S_0^*.

Since W is proper, $N(v) \cap (V(S_0) \setminus V(S_4)) \subseteq V(P)$. Let Q be the path from v to p_1 with $Q^* \subseteq V(P)$. Suppose first that $a_1 \notin V(S_4)$. Let S_5 be the sector of W with end a_1 and such that $V(S_5) \subseteq V(Q_1)$, and let b_3 be the other end of S_5. Since Q is shorter than P, it follows from the minimality of k that $V(S_4) \cap V(S_5) = \{b_3\}$ and $t = a_1$. Let R be the path from v to b_3 with $R^* \subseteq S_4^*$. Then x has exactly three neighbors in the hole $v - R - b_3 - S_5 - a_1 - p_1 - Q - v$, contrary to Lemma 8. This proves that $a_1 \in V(S_4)$.

Let b_3 be the other end of S_4, let S_5 be the second sector of W incident with b_3, and let a_3 be the other end of S_5. Since $v \notin X$, it follows that v has a neighbor $u \in V(G) \setminus V(W_1)$ such that u has a neighbor in $V(D_1) \setminus (V(S_4) \cup V(S_0))$. Since G is triangle-free, u is non-adjacent to x.

Suppose first that u has a neighbor in $V(Q_1) \setminus V(S_4)$. Since G is triangle-free and v has at least two neighbors in $V(P)$, it follows that $i \geq 4$, and therefore $k \geq 4$. Consequently, the path $u - v$ is shorter than P, and so it follows from the minimality of k that $N(u) \cap V(C) \subseteq V(S_5)$. Let R be the path from v to b_3 with $R^* \subseteq S_4^*$, and let D_2 be the hole $v - R - b_3 - x - v$. Let p be the neighbor of u in $V(S_5)$ closest to b_3, and let q be the neighbor of u in $V(S_5)$ closest to a_3. If $p \neq q$, we can link u to D_2 via the paths $u - p - S_5 - b_3$, $u - q - S_5 - a_3 - x$ and $u - v$, and if $p = q$ we can link p to D_2 via the paths $p - u - v$, $p - S_5 - b_3$ and $p - S_5 - a_3 - x$, in both cases contrary to Lemma 8. This proves that u has no neighbor in $V(Q_1) \setminus V(S_4)$, and therefore u has a neighbor in $V(T_0) \setminus V(S_0)$.

Next we define a new path Q. If u has a neighbor in $V(T_0) \cap V(S_2)$, let Q be the path $u - v$. If u is anticomplete to $V(T_0) \cap V(S_2)$, let j be maximum such that u is adjacent to p_j; then $j > i$; let Q be the path $v - u - p_j - P - p_k$. Since $i > 4$, in both cases $|V(Q)| < k$ and x has a unique neighbor in $V(Q)$. It follows from the minimality of k that $z = y = b_2 = a_3$. Since P violates the theorem, it follows that p_1 has a neighbor in $V(S_1) \setminus \{a_1\}$.

Let T be the path from v to a_1 with $T^* \subseteq V(S_4)$. Suppose that $s \neq t$. Let D_3 be the hole $x - a_1 - S_1 - s - p_1 - t - S_1 - b_1 - x$. Now v can be linked to D_3 via the paths $v - x$, $v - Q - p_1$ (short-cutting through a neighbor of b_1 if possible) and $v - T - a_1$, contrary to Lemma 8. Thus $s = t$, and therefore $s \neq a_1$. Bow we can link v to $x - S_1 - x$ via the paths $v - x$, $v - Q - p_1 - s$

(short-cutting through a neighbor of b_1 if possible) and $v - T - a_1$, contrary to Lemma 8. This proves (10).

In view of (10) let p_j be the unique neighbor of v in $V(P)$.

(11) *The fourth case of (9) holds.*

Suppose first that the case of the first bullet of (9) happens. Then by Lemma 8 v has at least four neighbors in the hole $x - S_0 - x$, and so, in view of (10), x has at least three neighbors in the path $a_1 - S_1 - s$, contrary to the fact that W is proper. By symmetry it follows that the cases of first two bullets of (9) do not happen. Suppose that the case of the third bullet of (9) happens. Since by (10) v has a unique neighbor in $V(P)$, it follows that v has a neighbor in $(V(S_0) \cup V(T_0)) \setminus V(P)$. By symmetry we may assume that v has a neighbor in $z - S_2 - b_2$, and, since W is proper, v is anticomplete to $V(S_0) \setminus V(P)$. Consequently, $p_j \in V(S_0) \setminus V(T_0)$, and so $j < i$. By the minimality of k (applied to the path $p_1 - P - p_j - v$), it follows that $j = k - 1$, and therefore $i = k$. Then $\{v, p_k\}$ is anticomplete to $V(C) \setminus V(S_2)$, since W is proper. By (5) v is adjacent to x. But now we get a contradiction to Lemma 10 applied to v and W_1. This proves (11).

In the next claim we further restrict the structure of P.

(12) *One of the following statements holds:*

- *there are edges between P^* and $V(C)$, or*
- *$j = 1$ and we can choose S_4 so that $a_1 \in V(S_4)$, or*
- *$j = k$ and we can choose S_4 so that $b_2 \in V(S_4)$.*

Suppose that (12) is false. Assume first that $j \notin \{1, k\}$. Then p_j is anticomplete to $V(C)$, since by assumption, there are no edges between P^* and C. If $s = t$, $y = z$ and v has a unique neighbor r in S_4, then $r \in V(S_4) \setminus (V(S_1) \cup V(S_2))$, and p_j can be linked to C via the paths $p_j - P - p_k - z$, $p_j - v - r$ and $p_j - P - p_1 - s$, contrary to Lemma 8. If some of p_1, p_k, v have several neighbors in C, then similar linkages work for the holes obtained from C by rerouting S_1 through p_1, S_2 through p_k, and S_4 through v, respectively. This proves that $j \in \{1, k\}$, and by symmetry may assume that $j = 1$. Then S_4 cannot be chosen so that $a_1 \notin V(S_4)$, for otherwise (12) holds. By the minimality of k and by (5), since S_4 cannot be chosen so that $a_1 \in V(S_4)$, it follows that x is adjacent to one of p_1, v and

$k = 2$. Since G is triangle-free, x has exactly one neighbor in $\{p_1, v\}$. Let R be the path from v to a_1 with $R^* \subseteq V(C) \setminus \{b_1\}$. Let Q'_1 be the subpath of R from an end of S_4 to a_1. Then $V(Q'_1) \subseteq V(Q_1)$ and $b_2 \notin V(Q'_1)$, and so the path $p_1 - v$ contradicts the choice of P. This proves (12).

The goal of the next two claims is to obtain more information about i and j.

(13) *$i = j$.*

Suppose not; by symmetry we may assume that $j < i$. Suppose first that x is non-adjacent to v. By (5) and the minimality of k, it follows that the first assertion of the theorem holds for the path $p_1 - P - p_j - v$; therefore $a_1 = t$ and S_4 can be chosen so that $a_1 \in V(S_4)$. Since W is proper it follows that v has at most two neighbors in S_4. If v has exactly two neighbors, then, in view of (6), v can be linked to $x - S_4 - x$ via two one-edge paths and the path $v - p_j - P - p_i - x$, contrary to Lemma 8. Therefore v has a unique neighbor r in S_4. Now, again in view of (6), p_j can be linked to $x - S_4 - x$ via the paths $p_j - v - r$, $p_j - P - p_1 - a_1$ and $p_j - P - p_i - x$, again contrary to Lemma 8. This proves that v is adjacent to x, and, since G is triangle-free, v has a neighbor in S_4^*. It follows that the choice of S_4 is unique. Let R be the path from v to a_1 with $R^* \subseteq V(C) \setminus \{b_1\}$. Suppose $a_1 \in V(S_4)$. Then $R^* \subseteq S_4^*$. In this case, because of (6) and since $b_1 \neq s$, p_j can be linked to the hole $v - R - a_1 - x - v$ via the path $p_j - v$, $p_j - P - p_1 - s - S_1 - a_1$ and $p_j - P - p_i - x$, contrary to Lemma 8. Thus $a_1 \notin V(S_4)$. Let S_5 be the sector of W with end a_1 such that $V(S_5) \subseteq V(Q_1)$. If $t = a_1$ and $V(S_4) \cap V(S_5) \neq \emptyset$, then x has exactly three neighbors in the hole $v - R - a_1 - p_1 - P - p_j - v$, contrary to Lemma 8. Therefore the path $p_1 - P - p_j - v$ violates the assertion of the theorem, and so the minimality of k implies that $j = k - 1$ and consequently $i = k$. Then by (6) a_2 is anticomplete to $V(P) \setminus \{p_k\}$. Since $j \neq k$ and $a_1 \notin V(S_4)$ (the choice of S_4 is now unique), it follows from (12) that there are edges between P^* and $V(C)$. Now by (6) there is a sector S_3 of W with ends a_2, b_1, and b_1 has a neighbor in P^*. Then there is a path T from b_1 to p_k with $T^* \subseteq P^*$, $b_1 - S_3 - a_2 - S_2 - y - p_k - T - b_1$ is a hole and x has exactly three neighbors in it, contrary to Lemma 8 (observe that $y \neq b_2$ because G has no triangles). This proves (13).

Since G is triangle-free, (13) implies that x is non-adjacent to v.

(14) *$i \leq 2$ and $i \geq k - 1$.*

Suppose (14) is false. By symmetry we may assume that $k - i > 1$. Consequently $k > 2$. Suppose that S_4 can be chosen so that $a_1 \in V(S_4)$. If v has a unique neighbor r in $V(S_4)$, then, since $s \neq b_1$, p_i can be linked to $x - S_4 - x$ via the paths $p_i - v - r$, $p_i - x$ and $p_i - P - p_1 - s - S_1 - a_1$, a contradiction. Thus v has at least two neighbors in $V(S_4)$. Now, again using the fact that $s \neq b_1$, p_i can be linked to the hole obtained from $x - S_4 - x$ by rerouting S_4 through v via the paths $p_i - v$, $p_i - x$ and $p_i - P - p_1 - s - S_1 - a_1$, again contrary to Lemma 8. Thus S_4 cannot be chosen so that $a_1 \in V(S_4)$. Let S_5 be the sector of W with end a_1 such that $V(S_5) \subseteq V(Q_1)$. Since $i \leq k - 2$, the minimality of k applied to the path $p_1 - P - p_i - v$ implies that $t = a_1$ and $V(S_4) \cap V(S_5) \neq \emptyset$. In particular, $i \neq 1$. It follows from (12) that there are edges between P^* and $V(C)$, and by (6) there is a sector S_3 of W with ends b_1, a_2 and every edge from p_i to $V(C)$ have an end in S_3^*. Together with the minimality of k (using the path $p_i - v$), this implies that p_i is anticomplete to $V(C)$. If v has a unique neighbor r in S_4 (and therefore $r \neq b_2$) and p_k has a unique neighbor in S_2, then p_i can be linked to C via the paths $p_i - v - r$, $p_i - P - p_1 - a_1$ (short-cutting through neighbors of b_1 if possible), and $p_i - P - p_k - z$ (short-cutting through neighbors of a_2 if possible). If v has at least two neighbors in $V(S_4)$ or or p_k has at least two neighbors in $V(S_2)$, then the same linkage works rerouting S_4 through v, and S_2 through p_k, respectively. This proves (14).

It follows from (13) and (14) that either

- $k = 3$ and $i = j = 2$, or
- $k = 2$.

If $k = 3$ and $i = j = 2$, then by (12) there are edges between P^* and $V(C)$, and so by (6) there is a sector S_3 with ends a_2, b_1, so that p_2 has neighbors in S_3^*; now the path $p_2 - v$ contradicts the minimality of k. Thus $k = 2$, and we may assume that $i = 1$, by symmetry. Since G is triangle-free, it follows that p_1 is non-adjacent to a_1, b_1. Since now $P^* = \emptyset$ is anticomplete to $V(C)$, it follows from (12) that we can choose S_4 with $a_1 \in V(S_4)$. Since v is non-adjacent to x and W is proper, it follows that v has at most two neighbors in S_4. If v has exactly two neighbors in S_4, then v can be linked to the hole $x - S_4 - x$ via two one-edge paths, and the path $v - p_1 - x$, contrary to Lemma 8. Thus v has a unique neighbor r in $V(S_4)$. Now p_1 can be linked to $x - S_4 - x$ via the paths $p_1 - v - r$, $p_1 - x$ and $p_1 - s - S_1 - a_1$, again contrary to Lemma 8. This completes the proof of Theorem 14. $\square$

We can now prove Theorem 6 which we restate:

Theorem 15. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free graph, and let x be the center of a proper wheel in G. If $W = (C, x)$ is a proper wheel with a minimum number of spokes subject to having center x, then*

1. *every component of $V(G) \setminus N(x)$ contains the interior of at most one sector of W, and*
2. *for every $u \in N(x)$, the component D of $V(G) \setminus (N(x) \setminus \{u\})$ such that $u \in V(D)$ contains the interiors of at most two sectors of W, and if S_1, S_2 are sectors with $S_i^* \subseteq V(D)$ for $i = 1, 2$, then $V(S_1) \cap V(S_2) \neq \emptyset$.*

Proof. To prove the first statement, we observe that if some component of $V(G) \setminus N(x)$ contains the interiors of two sectors of W, then this component contains a path violating the first assertions of Theorem 14.

For the second statement, suppose D contains the interiors of two disjoint sectors S_1, S_2 of W. Since $|D \cap N(x)| = 1$, we get a path in D violating the second assertion of Theorem 14. This proves Theorem 15. □

3 Proper Wheel Centers

In the proof of our main theorem, we perform manipulations on $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free graphs; in this section, we show that this preserves being $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free, and that no vertex becomes the center of a proper wheel.

Lemma 16. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free graph, $s \in V(G)$, K a component of $G \setminus N[s]$, and N the set of vertices in $N(s)$ with a neighbor in K. Let $H = G|(V(K) \cup N \cup \{s\})$. Then s is not the center of a proper wheel in H, and for $v \in V(H) \setminus \{s\}$, if v is the center of a proper wheel in H, then v is the center of a proper wheel in G.*

Proof. Since $H \setminus N[s]$ is connected, it follows that s is not the center of a proper wheel in H by Theorem 15. Let $v \in V(H) \setminus \{s\}$ be the center of a proper wheel $W = (C, v)$ in H. For all $w \in V(G) \setminus V(H)$, $N(w) \cap V(C) \subseteq N[s]$, and since G is triangle-free, it follows that every vertex $w \in V(G) \setminus V(H)$ either has at most one neighbor in $V(C)$, or $N(w) \cap V(C) \subseteq N(s)$.

Suppose that W is not proper in G. Then there exists a vertex w such that either w has more than two neighbors in a sector of W, but w is not adjacent to v, or w has neighbors in at least two sectors of W. It follows that w has more than one neighbor in $V(C)$, and thus in $N(s)$. Suppose that w has three distinct neighbors a, b, c in $V(C) \cap N(s)$. Let P be a shortest path connecting two of a, b, c, say a and b, with interior in K; then s is anticomplete to P^*. If c is anticomplete to P, then $G|(V(P) \cup \{w, s, a, b, c\})$ is an ISK_4. Otherwise, by the minimality of $|V(P)|$, P^* consists of a single vertex x, and $\{w, s, x, a, b, c\}$ induces a $K_{3,3}$ subgraph in G, a contradiction. So w has exactly two neighbors a and b in $V(C)$, and thus a and b are in different sectors of W. Since $a, b \in N(s)$ and W is proper in H, it follows that $s \in V(C)$ and s is a spoke of W; let S, S' be the two sectors of W containing s. But then v can be linked to the cycle $s-a-w-b-s$ via $v-s$ and the two paths with interiors in $S \setminus s$ and $S' \setminus s$. This is a contradiction by Lemma 8 and it follows that W is proper in G. This concludes the proof. □

We use the following well-known lemma, which we prove for completeness.

Lemma 17. *Let G be a connected graph, $a, b, c \in V(G)$ with $d(a) = d(b) = d(c) = 1$, and let H be a connected induced subgraph of G containing a, b, c with $V(H)$ minimal subject to inclusion. Then either H is a subdivision of $K_{1,3}$ with a, b, c as the vertices of degree one, or H contains a triangle.*

Proof. Let G, a, b, c, H be as in the statement of the theorem. Let P be a shortest $a-b$-path in H, and let Q be a shortest path from c to a vertex d with a neighbor in $V(P)$. By the minimality of $V(H)$, it follows that $V(H) = V(P) \cup V(Q)$. Moreover, P and Q are induced paths and no vertex of $Q \setminus d$ has a neighbor in $V(P)$. If d has exactly one neighbor in $V(P)$, then the result follows. If d has two consecutive neighbors in $V(P)$, then H contains a triangle. Otherwise, let $w \in V(P)$ such that d has a neighbor both on the subpath of P from w to a and on the subpath of P from w to b. It follows that $w \notin \{a, b, c\}$, and that $H \setminus w$ is connected and contains a, b, c. This contradicts the minimality of $V(H)$, and the result follows. □

Lemma 18. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free graph, s the center of a proper wheel in G, K a component of $G \setminus N[s]$, and N the set of vertices in $N(s)$ with a neighbor in K. Let G' arise from G by contracting $V(K)$ to a new vertex z. If $G|(V(K) \cup N \cup \{s\})$ is series-parallel, then G' is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free.*

Proof. G' does not contain a triangle, because $N_{G'}(z) \subseteq N_G(s)$ is stable, and hence z is not in a triangle in G'. Suppose that H is an induced subgraph of G' which is a $K_{3,3}$ or a subdivision of K_4. Then $z \in V(H)$. If z has degree two in H (and so H is an ISK_4), let a, b denote its neighbors; we can replace $a - z - b$ by an $a-b$-path P with interior in K and obtain a subdivision of H, which is an ISK_4, as an induced subgraph of G, a contradiction. Thus z has degree three in H; let a, b, c denote the neighbors of z in H. Let P be a shortest $a-b$-path with interior in K. Then c has at most one neighbor on P, for otherwise $G|(V(P) \cup \{a, b, c, s\})$ is a wheel, contrary to the fact that $G|(V(K) \cup N \cup \{s\})$ is series-parallel and does not contain a wheel by Theorem 3. Let Q be a shortest path from c to $V(P) \setminus \{a, b\}$ with interior in K; then each of a, b, c has a unique neighbor in $V(Q) \cup V(P)$ by symmetry. Let H' be a minimal connected induced subgraph of $G|(V(P) \cup V(Q))$ containing a, b, c. Since $G|(V(K) \cup N \cup \{s\})$ is series-parallel, it follows that H' is a subdivision of $K_{1,3}$ with a, b, c as the vertices of degree one by Lemma 17. Therefore, $G|(V(H \setminus z) \cup V(H'))$ is a subdivision of H, and by Theorem 4, it contains an ISK_4 or a $K_{3,3}$ subgraph in G. This is a contradiction, and the result is proved. □

Lemma 19. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free graph, s the center of a proper wheel in G, K a component of $G \setminus N[s]$, and N the set of vertices in $N(s)$ with a neighbor in K, and let $H = G|(V(K) \cup N \cup \{s\})$ be series-parallel. Let G' arise from G by contracting $V(K)$ to a new vertex z. Then z is not the center of a proper wheel in G', and for $v \in V(G') \setminus \{s, z\}$, if v is the center of a proper wheel in G', then v is the center of a proper wheel in G.*

Proof. Since $N_{G'}(z) \subseteq N_{G'}(s)$, it follows that z is not the center of a proper wheel in G', for otherwise s would have a neighbor in every sector of such a wheel. This proves the first statement of the lemma.

Throughout the proof, let $v \in V(G') \setminus \{s, z\}$ be the center of a proper wheel in G', and let $W = (C, v)$ be such a wheel with a minimum number of spokes. Since G' is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free by Lemma 18, it follows that W satisfies the hypotheses of Theorem 6. Our goal is to show that v is the center of a proper wheel in G.

(15) *If $z \in V(C)$, then v is the center of a proper wheel in G.*

Suppose that $z \in V(C)$. Let a, b denote the neighbors of z in $V(C)$. Let P be a shortest $a-b$-path with interior in K. Then every vertex in $V(K)$ has at most two neighbors in $V(P)$. Let $W' = (C', v)$ be the wheel in G that arises from W by replacing the subpath $a - z - b$ of C by $a - P - b$ to obtain C'.

It remains to show that W' is a proper wheel in G. Suppose that some vertex $x \in V(G) \setminus V(K)$ has two or more neighbors in P^*. Then $x \in N \subseteq V(H)$, and $H|(\{a, b, x, s\} \cup V(P))$ is a wheel in H with center x, a contradiction since H is series-parallel Theorem 3.

Since $v \notin V(K)$, it follows from the claim of the previous paragraph that v has at most one neighbor in P^*, and no neighbor unless v is adjacent to z, and therefore there are at most two sectors of W' intersecting P^*. We claim that if for a vertex x we have $|N_G(x) \cap V(C')| \geq 3$, then $x \notin V(K)$ and $|N_{G'}(x) \cap V(C)| \geq 3$. Suppose that x is a vertex violating this claim. If $x \in K$, then $N_G(x) \cap V(C') \subseteq V(P)$, and so $|N_G(x) \cap V(C')| \leq 2$ by the minimality of $|V(P)|$, a contradiction; it follows that $x \notin K$. Therefore, $|N_G(x) \cap P^*| \leq 1$, and thus $|N_G(x) \cap V(C')| - |N_{G'}(x) \cap V(C)| \leq 1$. But $|N_G(x) \cap V(C')| > 3$ by Lemma 8, and so $|N_{G'}(x) \cap V(C)| \geq 3$, a contradiction. So the claim holds.

Now suppose that there is a vertex x which is not proper for W'. If x has neighbors in at most one sector of W', then $|N_G(x) \cap V(C')| \geq 3$, but we proved above that $|N_{G'}(x) \cap V(C)| \geq 3$, and so, since W is proper, x is adjacent to v, a contradiction. It follows that x has neighbors in more than one sector of W'. Since x is proper for W, it follows that x has a neighbor in P^* and thus, either $x \in V(K)$ or x is adjacent to z. Since x is proper for W, it follows that $N_G(x) \cap V(C')$ is contained in the sectors of W' intersecting P^*. In particular, there are exactly two such sectors S_1 and S_2 of W', they are consecutive, and v has a neighbor in P^*. Consequently, v is adjacent to z and z is a spoke in W.

We claim that x has at most two neighbors in $V(C')$. If $x \in V(K)$ then $N_G(x) \cap V(C') \subseteq V(P)$ and we have already shown that every vertex of K has at most two neighbors in P. Thus we may assume that $x \notin K$, and so x is adjacent to z. Since G' is triangle-free by Lemma 18, it follows that x is not adjacent to v. Since x is proper for W, it follows that x has at most two neighbors in $V(C)$, and hence in $V(C')$, by our first claim. This proves our second claim. It follows that x has exactly one neighbor s_1 in $S_1 \setminus S_2$ and exactly one neighbor s_2 in $S_2 \setminus S_1$. If x is non-adjacent to v, then $G|(V(S_1) \cup V(S_2) \cup \{x, v\})$ is an ISK_4 in G, a contradiction. Therefore, x is adjacent to v and can be linked to the cycle $G|(V(S_1) \cup \{v\})$ via $x - v$, $x - s_1$, and a subpath of $x - s_2 - S_2$. Therefore W' is a proper wheel in G. This proves (15).

By (15), we may assume that $z \notin V(C)$. So W is a wheel in G. Since W is proper in G', there is a sector S of W containing all neighbors of z in C. Then clearly the following holds.

(16) *For every $x \in K$, $N_G(x) \cap V(C) \subseteq N_{G'}(z) \cap V(C) \subseteq V(S)$.*

Next we claim the following.

(17) *If z is not adjacent to v, then W is a proper wheel in G.*

If $x \in G \setminus (V(C) \cup V(K))$, then x is proper for W is in G as x is proper for W in G'. Now consider a vertex $x \in V(K)$. Since z is not adjacent to v, and W is proper in G', it follows that $|N_{G'}(z) \cap V(C)| \leq 2$. Then by (16), $|N_G(x) \cap V(C)| \leq 2$, and hence x is proper for W in G. This proves 17.

By (17), we may assume that z is adjacent to v. Let a and b be the ends of S. We now define a sequence of wheels in G with center v. Let $W_1 = W$ and $S_1 = S$. Assume that wheels $W_1, \dots, W_i$ have been defined, and define W_{i+1} as follows. If there is a vertex $x_i \in V(K)$ that is not adjacent to v and has at least three neighbors in S_i, then let S_{i+1} be the path from a to b in $G|(V(S_i) \cup \{x_i\})$ that contains x_i, and (by (16)) let W_{i+1} be the wheel obtained form W_i by replacing S_i by S_{i+1}. Since S_{i+1} is strictly shorter than S_i, this sequence must stop at some point; say it stops with wheel W_t. For $1 \leq i \leq t$, let C_i be the rim of W_i (so $V(C_i) = (V(C) \setminus V(S)) \cup V(S_i))$. Then $W_t = (C_t, v)$ is a wheel in G such that every vertex of K that has at least three neighbors in S_t is adjacent to v. We will show that W_t is a proper wheel in G, but first we show the following.

(18) *For $1 \leq i < t$, if a vertex y is proper for W_i, then y is proper for W_{i+1}.*

Suppose that y is proper for W_i and not proper for W_{i+1}. Then y is adjacent to x_i. Suppose first that y is non-adjacent to v and $|N_G(y) \cap V(C_{i+1})| \geq 3$. Since y cannot have three neighbors in C_{i+1} by Lemma 8, it follows that $|N_G(y) \cap V(C_{i+1})| > 3$. Moreover, since $N_G(y) \cap V(C_{i+1}) \subseteq \{x_i\} \cup (N_G(y) \cap V(C_i))$, it follows that $|N_G(y) \cap V(C_i)| \geq 3$. But then y is not proper for W_i, a contradiction. It follows that y has a neighbor in $C_{i+1} \setminus S_{i+1} = C \setminus S$, and thus $y \notin V(K)$. Therefore $y \in V(G')$, and since y is adjacent to x_i in G, it follows that y is adjacent to z in G'. Since z is adjacent to v in G' and G' is triangle-free by Lemma 18, it follows that y is non-adjacent to v. Note that since $i + 1 > 1$, it follows that a vertex of K has a neighbor in S^*, and therefore z has a neighbor in S^*. Since W satisfies the hypotheses of Theorem 6, and since $y - z$ is a path containing exactly one neighbor of v, it follows that the neighbors of y in C are in a sector S' of W consecutive with S. Since y is non-adjacent to v, it follows that y has at most two neighbors in S'. Note that since G' is triangle-free, and $N_{G'}(z) \cap V(C) \subseteq V(S)$, it follows that z has no neighbors in S'. If y has exactly two neighbors in S', then y can be linked in G' to the hole $v - S' - v$ via two one-edge paths and the path $y-z-v$. So y has exactly one neighbor r in S' that is in $V(S') \setminus V(S)$, and now z can be linked to $v - S' - v$ via the paths $z - v$, $z - y - r$, and a path with interior in S, contrary to Lemma 8. This concludes the proof of (18).

Every vertex in $G \setminus V(K)$ is proper for W and hence it is proper for W_t by (18). Suppose that there is a vertex $x \in V(K)$ that is not proper for W_t. By (16), $N_G(x) \cap V(C_t) \subseteq V(S_t)$. So x is non-adjacent to v and has at least three neighbors in S_t, contradicting the assumption that the wheel sequence terminates with W_t. Therefore, W_t is a proper wheel in G with center v.

□

4 Tools

In this section we develop tools for our main theorem for finding a vertex of degree one, or a cycle with all but a few vertices of degree two.

Lemma 20. *Let G be a graph, $x \in V(G)$, such that $G \setminus x$ is a forest. Then either $V(G) = N[x]$ and $G \setminus x$ is stable, or $V(G) \setminus N[x]$ contains a vertex of degree at most one in G, or G contains an induced cycle C containing x such that every vertex of $V(C) \setminus \{x\}$ except for possibly one has degree two in G.*

Proof. If every component of $G \setminus x$ contains exactly one vertex, then either $V(G) = N[x]$ or $V(G) \setminus N[x]$ contains a vertex of degree zero. Hence, we may assume that there exists a component T of $G \setminus x$ with at least two vertices, and T is a tree. Let A be the set of vertices of degree at least three in T. If A is non-empty, then let T' be the subtree of T that contains all vertices of A and minimal with respect to this property, and let a be a leaf of T'. There is a path $P = v - \cdots - v'$ in T, whose ends are distinct leaves of T and P contains at most one vertex of degree three in T (namely a). This is trivial is A if empty, and follows from the definition of a otherwise.

If x is non-adjacent to v, then v is a vertex in $V(G) \setminus N[x]$ of degree one in G, so we may assume that x is adjacent to v, and similarly for v'. Now, let v'' be the neighbor of x in $P \setminus v$ closest to v along P. We set $C = x - v - P - v'' - x$ and observe that all vertices of C except possibly x and a have degree two in G. □

Lemma 21. *Let G be a series-parallel graph, and let $x, y \in V(G)$ with $x = y$ or $xy \in E(G)$. If $G \setminus \{x, y\}$ contains a cycle, then there is an induced cycle C*

in G such that $V(C) \cap \{x, y\} = \emptyset$ and all but at most two vertices of C have degree two in G (and are thus anticomplete to $\{x, y\}$), or $V(G) \setminus (N[x] \cup N[y])$ contains a vertex of degree at most one in G.

Proof. By contracting the edge xy and deleting any parallel edges that may arise, we may assume that $x = y$. We may further assume that every vertex except for possibly x has degree at least two, because vertices of degree one in $N(x)$ can be deleted without affecting the hypotheses or the conclusions, and if there is a vertex of degree at most one in $V(G) \setminus N[x]$, then the conclusion holds.

Let C be a cycle in $G \setminus x$. Since G is series-parallel and by the definition of series-parallel graphs, it follows that there do not exist three paths from x to $V(C)$ that are vertex disjoint except for x in G. By Menger's theorem [4], it follows that there is a partition (X, Y, Z) of $V(G)$ with X of size at most two, and $Y, Z \neq \emptyset$ such that Y is anticomplete to Z in G, $V(C) \subseteq Y \cup X$ and $x \in Z$.

We choose a partition (X, Y, Z) with $|X|$ minimal, and subject to that, $|X \cup Y|$ minimal, such that Y is anticomplete to Z in G, $Y, Z \neq \emptyset$, $x \in Z$, and $G|(Y \cup X)$ contains a cycle. It follows that $|X| \leq 2$.

Suppose first that $X = \emptyset$. If $G|Y$ is an induced cycle, the result follows. Otherwise, since $G|Y$ contains a cycle, it follows that there is a vertex x' such that $G|(Y \setminus \{x'\})$ contains a cycle. By induction applied to $G|Y$ and the vertex x', the result follows.

Next, suppose that $X = \{x'\}$. If $G|Y$ is a forest, then $x' \neq x$ and thus we obtain the desired result by applying Lemma 20 to $G|(X \cup Y)$. Otherwise, we apply induction to $G|(X \cup Y)$ and x', and again, the result follows.

It follows that $X = \{x', y'\}$, and therefore, the component of $G|(Z \cup X)$ containing x contains x' and y', for otherwise $\{x'\}$ or $\{y'\}$ would be a better choice of X for the partition. Suppose that $G|Y$ is connected. If there is a vertex z such that every $x' - y'$-path with interior in $G|Y$ uses z, then $\{x', z\}$ or $\{y', z\}$ yields a better choice of X and partition. Therefore, by Menger's theorem [4], there are two disjoint paths P_1, P_2 from x' to y' with interior in Y, and since $G|Y$ is connected, it follows that there is path Q from P_1 to P_2 in Y. Moreover, there is a path R from x' to y' with interior in Z since the component of $G|(Z \cup X)$ containing x also contains x' and y'; but $P_1 \cup P_2 \cup Q \cup R$ is a (not necessarily induced) subdivision of K_4 in G, contrary to the fact that G is series-parallel. Thus $G|Y$ is not connected. By the minimality of $X \cup Y$, for every component K of $G|Y$, the graph $G|((X \setminus \{x\}) \cup V(K))$ contains no cycle. However, $G|(X \cup Y)$ contains a cycle C not using x, and so C contains vertices from more than one component of $G|Y$. It follows that $x', y' \in V(C)$, and thus $x \notin \{x', y'\}$. Therefore, for every component K of $G|Y$, the graph $G|(X \cup V(K))$ is a tree. Since K is connected, it follows that x', y' are leaves. If $G|(X \cup V(K))$ contains a leaf other than x', y', then the result follows. So each component is a path from x' to y', and no vertex of the path except for x', y' has further neighbors in G. But then the union of two of those paths (there are at least two, since $G|Y$ is not connected) yields the desired cycle; the result follows. □

Theorem 22. *Let G be an $\{\mathrm{ISK}_4, \text{triangle}, K_{3,3}\}$-free graph, $x, y \in V(G)$ with $x = y$ or $xy \in E(G)$. Then either*

- *$V(G) = N[x] \cup N[y]$;*
- *there exists a vertex in $V(G) \setminus (N[x] \cup N[y])$ of degree at most one in G;*
- *there exists an induced cycle C containing at least one of x, y such that at most one vertex v in $V(C) \setminus (N[x] \cup N[y])$ has $d(v) > 2$; or*
- *there exists an induced cycle C containing neither x nor y and a vertex $z \in V(C)$ such that at most one vertex v in $V(C) \setminus N[z]$ has $d(v) > 2$.*

Proof. Suppose first that G is series-parallel. Define $H = G$ and $v = x$ if $x = y$, and define H as the graph that arises from contracting the edge xy to a new vertex v if $x \neq y$. Then H is series-parallel. Suppose that $H \setminus v$ is a forest, and apply Lemma 20. If the first outcome of Lemma 20 holds, then $V(H) = N_H(v)$, and so $V(G) = N_G(x) \cup N_G(y)$. If the second outcome of Lemma 20 holds, then $V(H) \setminus N_H[v]$ contains a vertex of degree at most one in H, and so $V(G) \setminus (N_G[x] \cup N_G[y])$ contains a vertex of degree at most one in G. Finally, if the third outcome of Lemma 20 holds, then H contains an induced cycle C containing v such that every vertex of $V(C) \setminus \{v\}$ except for at most one has degree two in H, and so there is an induced cycle C' in G containing at least one of x, y such that every vertex of $V(C') \setminus \{x, y\}$ except for possibly one has degree two in G. This proves the result in the case that $H \setminus v$ is a forest. So $H \setminus v$ contains a cycle, and thus $G \setminus \{x, y\}$ contains a cycle. By Lemma 21, either $V(G) \setminus (N[x] \cup N[y])$ contains a vertex of degree at most one in G, or G contains a cycle C with $V(C) \cap \{x, y\} = \emptyset$ and such that all but at most two vertices in $V(C)$ have degree two in G. In the former case, the second outcome of this theorem holds; in the latter case, the fourth outcome of this theorem holds by choosing $z \in V(C)$ with $d_G(z)$ maximum among vertices in $V(C)$.

Thus we may assume that G contains a proper wheel by Lemma 9; let z be the center of a proper wheel (where possibly $z \in \{x, y\}$). Let W be such a wheel with minimum number of spokes. Let $Z = \{x, y\} \cap N(z)$. Since $x = y$ or $xy \in E(G)$, it follows that x and y are in the same component of $G \setminus (N[z] \setminus Z)$. Since $N(z)$ is stable, it follows that $|Z| \leq 1$. Therefore, by Theorem 6, the component of $G \setminus (N[z] \setminus Z)$ containing $\{x, y\} \setminus \{z\}$ includes the interiors of at most two sectors of W. Again by Theorem 6, the interior of every other sector of W is contained in a separate component of $G \setminus (N[z] \setminus Z)$. Since W has at least four sectors by Lemma 8, there is a component K of $G \setminus (N[z] \setminus Z)$ that does not contain x and y, and that contains no neighbor of x, y. Let N be the set of neighbors of z with a neighbor in K. Then, we apply induction to $H = G|(V(K) \cup N \cup \{z\})$ and z. By the choice of H and z, the first outcome does not hold. If the second outcome holds for H and z, then it holds for G and x, y as well, since $(N[x] \cup N[y]) \cap V(H) \subseteq N[z] \cap V(H)$. If the third or fourth outcome holds for H and z, then the third or fourth outcome holds for G and x, y. □

5 Main Result

We say that (G, x, y) has the property $\mathcal{P}$ if $V(G) \setminus (N[x] \cup N[y])$ contains a vertex of degree at most two in G.

We can now prove Theorem 7 which we restate:

Theorem 23. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free graph which is not series-parallel, and let (x, y) be a non-center pair for G. Then (G, x, y) has the property $\mathcal{P}$.*

Proof. Suppose for a contradiction that the theorem does not hold, and let (G, x, y) be a counterexample with $|V(G)|$ minimum. Then every vertex in $V(G) \setminus (N[x] \cup N[y])$ has degree at least three in G. Since G is not series-parallel, and G is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free, it follows from Theorem 4 that G contains a wheel and hence by Lemma 9, it follows that G contains a proper wheel $W = (C, s)$. Let $C_1, \dots, C_k$ denote the components of $V(G) \setminus N[s]$. For $i = 1, \dots, k$ let N_i denote the set of neighbors v of s such that v has a neighbor in C_i, and let G_i denote the induced subgraph of G with vertex set $V(C_i) \cup N_i \cup \{s\}$.

(19) *For $i = 1, \dots, k$, if G_i is series-parallel and $G_i \setminus (N[s] \cap \{x, y, s\})$ contains a cycle, then $\{x, y\} \cap V(C_i) \neq \emptyset$.*

Let $i \in \{1, \dots, k\}$ such that G_i is series-parallel, and let $G_i \setminus (N[s] \cap \{x, y, s\})$ contain a cycle. Since G is triangle-free, it follows that $1 \leq |N[s] \cap$

$\{x, y, s\} \mid \leq 2$. By Lemma 21 applied to G_i and the vertices in $N[s] \cap \{x, y, s\}$, it follows that either there is a vertex in $V(G_i) \setminus N[s]$ of degree at most one in G_i anticomplete to $\{x, y\} \cap N(s)$, or $G_i \setminus (N[s] \cap \{x, y, s\})$ contains a cycle C' with at least two vertices of degree two in G_i. In both cases, there is a vertex z in $V(G_i) \setminus N[s]$ of degree at most two in G_i and z is anticomplete to $N[s] \cap \{x, y, s\}$, and hence its degree in G is also at most two. Since (G, x, y) does not satisfy property $\mathcal{P}$, it follows that $z \in N[x] \cup N[y]$, and thus $\{x, y\} \cap V(C_i) \neq \emptyset$. This proves (19).

(20) *For $i = 1, \ldots, k$, if G_i is series-parallel, then $|V(C_i)| = 1$.*

Let $i \in \{1, \ldots, k\}$ be such that G_i is series-parallel, and suppose that $|V(C_i)| > 1$. Let G' be the graph that arises from G by contracting $V(C_i)$ to a new vertex z. We let $x' = z$ if $x \in V(C_i)$ and $x' = x$ otherwise; and we let $y' = z$ if $y \in V(C_i)$ and $y' = y$ otherwise. By Lemma 18, G' is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free. By Lemma 19, (x', y') is a non-center pair for G'. By the minimality of $|V(G)|$, it follows that (G', x', y') has the property $\mathcal{P}$. Let $v \in V(G') \setminus (N[x'] \cup N[y'])$ be a vertex of degree at most two in G'. From the definition of x' and y', it follows that $v \notin N[x] \cup N[y]$. It follows that either $v = z$, or $v \neq z$ and $d_G(v) > 2$, and so $v \in N[z]$.

Suppose first that $v = z$. Then $z \notin N[x'] \cup N[y']$, and so $V(G_i) \cap \{x, y\} = \emptyset$. By (19), it follows that $G_i \setminus s$ is a tree. Since v has degree at most two in G', it follows that $|N_i| \leq 2$, and since $G_i \setminus N[s]$ is connected, it follows that every vertex of N_i is a leaf of $G_i \setminus s$. Thus, either $V(C_i)$ contains a leaf of $G_i \setminus s$, or $G_i \setminus s$ is a path with ends in N_i, and so in both cases $V(C_i)$ contains a vertex of degree at most two in G. This is a contradiction since $V(G_i) \cap \{x, y\} = \emptyset$; it follows that $v \neq z$.

It follows that $v \in N(z)$. Since $d_G(v) > 2$, it follows that $d_{G'}(v) < d_G(v)$, and thus v has more than one neighbor in $V(C_i)$. Let P be a path in C_i between two neighbors of v, then $v - P - v$ is a cycle in $G_i \setminus (N[s] \cap \{x, y, s\})$. By (19), it follows that $V(C_i) \cap \{x, y\} \neq \emptyset$. But then $z \in \{x', y'\}$, and so $v \in N[x'] \cup N[y']$, a contradiction. This proves (20).

(21) *For $i = 1, \ldots, k$, if G_i contains a wheel, then $x \in V(C_i)$ or $y \in V(C_i)$.*

Suppose not, and let $i \in \{1, \ldots, k\}$ be such that G_i contains a wheel and $V(C_i) \cap \{x, y\} = \emptyset$. Since N_i is a stable set, it follows that $|N_i \cap \{x, y\} \mid \leq 1$, and by symmetry, we may assume that $y \notin N_i$. Let $y' = s$, and let $x' = x$ if $x \in N_i$ and $x' = s$ otherwise. By Lemma 16, (x', y') is a non-center pair for G_i. Since G_i is an induced subgraph of G, it follows that G_i is

$\{\mathrm{ISK}_4, \text{triangle}, K_{3,3}\}$-free. Since G is a minimum counterexample, it follows that (G_i, x', y') has the property $\mathcal{P}$. Let v be a vertex of degree at most two in G_i with $v \notin N[x'] \cup N[y']$. Since $v \notin N[s]$, it follows that $d_G(v) = d_{G_i}(v)$. Therefore $v \in N[x] \cup N[y]$. Let $w \in \{x, y\}$ be such that $v \in N[w]$. By assumption, $w \notin V(C_i)$. If $w \in V(G_i)$, then $w \in N_i$, and so $w = x = x'$, and thus $v \notin N[w]$, a contradiction. It follows that $w \notin V(G_i)$, and so $N[w] \cap V(G_i) \subseteq N[s]$, and again $v \notin N[w]$, a contradiction. This proves that $\{x, y\} \cap V(C_i) \neq \emptyset$, and (21) follows.

It follows from Theorem 3 together with (21) and (20) that there is at most one $i \in \{1, \dots, k\}$ with $|V(C_i)| > 1$. We may assume $|V(C_i)| = 1$ for all $i \in \{1, \dots, k-1\}$.

(22) *If $|V(C_k)| > 1$, let $G' = G \setminus (V(C_k) \cup \{s\})$; otherwise let $G' = G \setminus s$. Then G' has girth at least eight.*

Observe that G' is bipartite with one side of the bipartition being $N(s)$.

Suppose that C is a cycle of length four in G'. Let $V(C) = \{a, b, c, d\}$ and $N(s) \cap V(C) = \{a, c\}$. If $d_G(b) \neq 2$, let e be a neighbor of b which is not a, c. Note that $e \in N(s)$. Then $\{a, b, c, d, e, s\}$ induces an ISK_4 or a $K_{3,3}$, a contradiction. It follows that $d_G(b) = d_G(d) = 2$, and moreover, $\{x, y\} \cap \{a, c\} \neq \emptyset$. So, by symmetry, say $x = a$, and we may assume that $d \neq y$.

Observe that $G \setminus b$ is not series-parallel by Theorem 3. By the minimality of $|V(G)|$, it follows that there exists $v \in V(G) \setminus (N[x] \cup N[y])$ with $d_{G\setminus\{b\}}(v) \leq 2$. Since $d_G(v') = d_{G\setminus\{b\}}(v')$ for all $v' \in V(G) \setminus \{x, b, c\}$, it follows that $v = c$, and so $N_G(c) = \{b, d, s\}$, and so $\{s, a\}$ is a cutset in G. Let $G'' = G \setminus \{b, c, d\}$, and if $y \in \{b, c, d\}$, let $y' = x$, otherwise, $y' = y$. Then G'' is not series-parallel. A proper wheel in G'' is proper in G, because each vertex in $\{b, c, d\}$ has at most one neighbor in the wheel, s or a. Therefore, (x, y') is a non-center pair for G''. By the minimality of $|V(G)|$, it follows that (G'', x, y') has the property $\mathcal{P}$. But this is a contradiction, since every vertex in $V(G'') \setminus N[x]$ has the same degree in G and G''. This proves that G' contains no 4-cycle.

Suppose G' contains a 6-cycle C. Then, since exactly three vertices in $V(C)$ are neighbors of s, it follows that $G|(V(C) \cup \{s\})$ is an ISK_4, a contradiction. It follows that G' has girth at least eight, and so (22) is proved.

(23) $|V(C_k)| > 1$.

Suppose not, and let $G' = G \setminus s$. Then G' satisfies the hypotheses of Theorem 22. Since s is the center of a proper wheel in G, it follows that there exists a vertex z in G' that is not in $N[x] \cup N[y] \cup N[s]$, and so the first outcome of Theorem 22 does not hold. The second outcome does not hold, because every vertex in $V(G') \setminus (N[x] \cup N[y])$ of degree one in G' has degree at most two in G, a contradiction.

Therefore, the third or fourth outcome of Theorem 22 holds, and hence there exists an induced cycle C in G' with vertices $c_1 - \ldots - c_t - c_1$, and $i, j \in \{1, \ldots, t\}$, $l \in \{0, \ldots, 3\}$ such that all vertices of C except for $c_i, \ldots, c_{i+l}$ (where $c_{t+1} = c_1$ and so on) and c_j have degree two in G', do not coincide with x, y and are non-neighbors of x, y. By (22), $t \geq 8$. Consequently, G' contains two adjacent vertices in $V(G') \setminus (N[x] \cup N[y])$ of degree two in G'. Since G is triangle-free, it follows that one of them has degree two in G, a contradiction. Thus, $|V(C_k)| > 1$, and (23) is proved.

By (20), (21) and (23) we may assume that $x \in V(C_k)$. Let G' arise from G by contracting $V(C_k) \cup N_k$ to a single vertex z, and by deleting s and every vertex that is only adjacent to z. It follows that G' is bipartite. Our goal is to prove that $G' \setminus z$ has girth at least 16, see (28). By (22), we know that $G' \setminus z$ has girth at least eight.

(24) *Every vertex in $V(G') \setminus \{z\}$ has at most one neighbor in N_k in G. There is no 4-cycle in G' containing z.*

Suppose first that there is a vertex $v \in V(G') \setminus \{z\}$ with at least two neighbors $a, b \in N_k$ in G. Since $v \in V(G')$ and in G' there are no vertices of degree one adjacent to z, it follows that v has another neighbor $c \in N(s) \setminus N_k$. Let P be a path connecting a and b with interior in $V(C_k)$. Such a path exists, since $a, b \in N_k$. It follows that $G|(V(P) \cup \{a, b, c, v, s\})$ is an ISK_4 in G, a contradiction. This implies the first statement of (24).

Suppose that z is contained in a 4-cycle with vertex set $\{a, b, c, z\}$ in G' such that $a, c \in N_{G'}(z)$. Note that $a, c \notin N(s)$ and $b \in N(s) \setminus N_k$. By (22), $G \setminus (\{s\} \cup V(C_k))$ contains no 4-cycle, and thus a and c have no common neighbor in N_k. Let a', c' be a neighbor of a and c in N_k, respectively; a' and c' exists since $a, c \in N_{G'}(z)$. Let P be a shortest path between a' and c' with interior in C_k. Since $b \notin N_k$, it follows that b is anticomplete to $V(P)$. Therefore, $G|(\{a, b, c, s\} \cup V(P))$ is an ISK_4 in G, a contradiction. This proves (24).

(25) G' *is* $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$*-free.*

Since G' is bipartite, it follows that G' is triangle-free. Suppose that G' contains an induced subgraph H which is either a $K_{3,3}$ or an ISK_4. Since G is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free, it follows that $z \in V(H)$. Suppose that z has degree two in H. By (24), the neighbors of z in $V(H)$ do not have a common neighbor in N_k. Let P be a path in G connecting the neighbors of z in $V(H)$ with interior in $V(C_k) \cup N_k$ containing exactly two vertices in N_k. Then $G|((V(H) \setminus \{z\}) \cup V(P))$ is an induced subdivision of H in G. By Theorem 4, it follows that G is not $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free, a contradiction.

It follows that z has degree three in H. Let a, b, c be the neighbors of z in H. By (24), each of a, b, c has a unique neighbor in N_k. Let a', b', c' be neighbors of a, b, c in N_k. Let H' be a minimal induced subgraph of $G|(V(C_k) \cup \{a, b, c, a', b', c'\})$ which is connected and contains $\{a, b, c\}$. It follows that each of a, b, c has a unique neighbor (namely a', b', c', respectively), in H'. By Lemma 17, H' is a subdivision of $K_{1,3}$ in which a, b, c are the vertices of degree one. Consequently, $G|(V(H \setminus z) \cup V(H'))$ is an induced subgraph of G which is a subdivision of H. But then G is not $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free by Theorem 4. Hence G' is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$-free. This proves (25).

(26) *G' does not contain a proper wheel with center different from z.*

Suppose $v \neq z$ is the center of a proper wheel G'. By Theorem 6, there is a component C of $G' \setminus N[v]$ that is disjoint from $N[z]$. Let N denote the set of vertices in $N(v)$ with a neighbor in C.

Then $H = G'|(N \cup V(C) \cup \{v\})$ satisfies the hypotheses of Theorem 22. Since $V(C) \neq \emptyset$, it follows that the first outcome of Theorem 22 does not hold. Moreover, every vertex in $V(H) \setminus N[v]$ of degree one in H has degree at most two in G, since such a vertex belongs to C and C is disjoint from $N[z]$, and the only additional neighbor that such a vertex may have in G is s. Furthermore, such a vertex is in $V(G') \setminus N[z]$, and hence in $V(G) \setminus (N[x] \cup N[y])$ as $x \in V(C_k)$. It follows that the second outcome of Theorem 22 does not hold.

Therefore, the third or fourth outcome of Theorem 22 holds, and hence there exists an induced cycle C' in H with vertices $c_1 - \ldots - c_t - c_1$, and $i, j \in \{1, \ldots, t\}$, $l \in \{0, \ldots, 3\}$ such that all vertices of C' except for $c_i, \ldots, c_{i+l}$ (where $c_{t+1} = c_1$ and so on) and c_j have degree two in H, do not coincide with v and are non-neighbors of v. By (22), $t \geq 8$, since $z \notin V(H)$. Consequently, G' contains two adjacent vertices in $V(G') \setminus N[z]$ of degree two in G'. Since G is triangle-free, it follows that one of them is non-adjacent to s and thus has degree two in G, a contradiction. Hence (26) is proved.

(27) *For every component K of $G' \setminus N[z]$, $G'|(V(K) \cup N(z))$ is a forest.*

Suppose not, and let K be a component of $G' \setminus N[z]$ such that $G'|(V(K) \cup N(z))$ is not a forest. Suppose first that $H = G'|(V(K) \cup N[z])$ is not series-parallel. Then H contains a proper wheel by Lemma 9. Let v be the center of a proper wheel in H. Since $H \setminus N[z]$ is connected, it follows from Theorem 6 that $v \neq z$. By Lemma 16, it follows that v is the center of a proper wheel in G', contrary to (26).

It follows that H is series-parallel, and by our assumption, $H \setminus z$ contains a cycle. By applying Lemma 21 to H and z, it follows that there is either a vertex in $V(H) \setminus N[z]$ of degree one, or a cycle C not containing z, with all but at most two vertices of degree two in H. In the latter case, since $G' \setminus z$ has girth at least eight, C contains two adjacent vertices in $V(H) \setminus N[z]$ of degree two in H, and thus of degree two in G'. Since G is triangle-free, it follows that in both cases G contains a vertex of degree at most two not in $N[z]$, and thus not in $N[x] \cup N[y]$. This is a contradiction, and (27) is proved.

(28) *The girth of $G' \setminus z$ is at least 16.*

Suppose that this is false. Let C be an induced cycle in $G' \setminus z$ of length less than 16. Since by (27), for every component K of $G' \setminus N[z]$, we have that $G'|(V(K) \cup N(z))$ is a forest, it follows that $C \setminus N[z]$ has at least two components. Since z is not contained in a 4-cycle in G' by (24), and G' is bipartite, it follows that each component of $C \setminus N[z]$ has at least three vertices. If $C \setminus N[z]$ has at least four components, it follows that C has length at least 16. If $C \setminus N[z]$ has exactly three components, then $G|(V(C) \cup \{z\})$ is an ISK$_4$, a contradiction. So $C \setminus N[z]$ has exactly two components. For every component K of $G' \setminus N[z]$, by (27) we have that $G'|(V(K) \cup N(z))$ is a forest. Therefore, the two components of $C \setminus N[z]$ are contained in two different components of $G' \setminus N[z]$; say A and B. Let N_A, N_B denote the set vertices in $N(z)$ with a neighbor in A, B, respectively. Suppose that $|N_A| \geq 3$. Since $|V(C) \cap N(z)| = 2$, it follows that there is a path P from a vertex c in $N_A \setminus V(C)$ to $V(C)$ with interior in $V(A)$. Since $G'|(V(A) \cup N_A)$ and $G'|(V(B) \cup N_B)$ are trees, it follows that c has at most one neighbor in each component K of $C \setminus N[z]$. Therefore, $G'|(V(P) \cup V(C) \cup \{z\})$ contains an induced subgraph of G' which is either a subdivision of K_4 or of $K_{3,3}$, a contradiction by Theorem 4 and (25). So $|N_A| = 2$. Since $G'|(V(A) \cup N_A)$

is a tree, it follows that either A contains a vertex of degree one in G', non-adjacent to z, or $G'|(V(A) \cup N_A)$ is a path containing at least five vertices, and hence A contains two adjacent vertices of degree two in G', non-adjacent to z. Since G is triangle-free, it follows that in either case G contains a vertex of degree at most two not in $N[z]$, and thus not in $N[x] \cup N[y]$. This is a contradiction, and (28) is proved.

Recall that $\{x, y\} \cap V(C_k) \neq \emptyset$, and we may assume that $x \in V(C_k)$, and thus $y \in V(C_k) \cup N_k$. Let G'' be the graph that arises from G by deleting $\{s\} \cup (V(C_k) \setminus \{x\}) \cup (N_k \setminus \{y\})$, and every vertex other than x with neighbors only in N_k (this last operation does not change the degree of any vertex in $V(G'')$ except for possibly y). Then $N_{G''}(x) \subseteq \{y\}$. It follows from (28) that $G'' \setminus \{y\}$ has girth at least 16, and from (22) that G'' has girth at least eight. If $y \in V(G'')$, let $y' = y$; otherwise, let $y' = x$. It follows that if $y' = y$, then $y \in N_k$.

Since G'' is an induced subgraph of G, it follows that G'' and x, y' satisfy the hypotheses of Theorem 22.

Since s is the center of a proper wheel, it follows from Theorem 6 that there are at least two components of $G'' \setminus N[s]$ in which y' has no neighbors. Consequently, $V(G'') \neq N[x] \cup N[y']$, and thus the first outcome of Theorem 22 does not hold.

The second outcome of Theorem 22 does not hold, because if G'' contains a vertex v of degree one non-adjacent to y', then v has degree at most two in G, and $v \notin N[x] \cup N[y]$, a contradiction.

Suppose that the third outcome holds, and so G'' contains an induced cycle C containing y' (since $d_{G''}(x) \leq 1$) such that at most one vertex in $V(C) \setminus N[y]$ has degree more than two. Since G'' has girth at least eight, $|V(C)| \geq 8$, and in particular C contains a vertex v of distance three from y in C and degree two in G''. Let $y - a - b - v$ be the three-edge path from y to v in C. Then v is not adjacent to s in G, because $G|(V(G'') \cup \{s\})$ is bipartite and $ys \in E(G)$. Moreover, v anticomplete to N_k, because otherwise $z - a - b - v - z$ is a 4-cycle in G' using z, contradicting (24). So v has degree two in G and is not in $N[x] \cup N[y]$, a contradiction.

Thus, the fourth outcome holds, and so G'' contains an induced cycle C not containing x, y' and containing a vertex z' such that at most one vertex in $V(C) \setminus N[z']$ has degree more than two in G''. Since $|V(C)| \geq 16$, it follows that C contains a path $P = p_1 - \ldots - p_6$ of six vertices, all of degree two in G'' and non-adjacent to x, y'. We may assume that $N(s) \cap V(P) \subseteq \{p_1, p_3, p_5\}$ by symmetry. Since z is not in a 4-cycle in G' by (24), not both p_2 and p_4

have a neighbor in N_k. It follows that either p_2 or p_4 has degree two in G, a contradiction. This completes the proof of Theorem 23. □

We can now prove Theorem 2 which we restate:

Theorem 24. *Let G be an* $\{\text{ISK}_4, \text{triangle}\}$*-free graph. Then either G has a clique cutset, G is complete bipartite, or G has a vertex of degree at most two.*

Proof. Let G be an $\{\text{ISK}_4, \text{triangle}\}$-free graph. If G is series-parallel, then G contains a vertex of degree at most two Theorem 3. If G contains $K_{3,3}$ as a subgraph, then by Theorem 5, either G is complete bipartite or G has a clique cutset. If G is not series-parallel and $K_{3,3}$-free, then G contains a vertex of degree at most two by Theorem 7 applied to the graph obtaining from G by adding an isolated vertex x with the non-center pair (x, x). This implies the result. □

Note that the outcome of a clique cutset in Theorem 24 cannot be avoided, as the following example shows. Let G be any $\{\text{ISK}_4, \text{triangle}\}$-free graph (e. g. a C_5, or a wheel), and let H arise from G by adding $|V(G)|$ disjoint copies of $K_{3,3}$ to G and identifying each vertex of G with a vertex of a different copy of $K_{3,3}$. The resulting graph is $\{\text{ISK}_4, \text{triangle}\}$-free, not series-parallel, and not bipartite if G is not bipartite, and it contains no vertex of degree at most two.

Finally we are ready to prove the following.

Theorem 25. *If G is an* $\{\text{ISK}_4, \text{triangle}\}$*-free graph, then G is 3-colorable.*

Proof. The proof is by induction on $|V(G)|$ using Theorem 2. If G is complete bipartite, then G is 2-colorable. If G has a vertex v of degree at most two, then, by induction, $G \setminus v$ is 3-colorable, and hence G is 3-colorable. If G has a clique cutset C such that (A, B, C) is a partition of $V(G)$ with A anticomplete to B and C a clique, then $\chi(G) = \max\{\chi(G|(A \cup C)), \chi(G|(B \cup C))\}$, and again by induction, G is 3-colorable. □

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