Obstructions for three-coloring and list three-coloring H-free graphs

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In loving memory of Ella.

Abstract

We characterize all graphs H for which there are only finitely many H-free 4-vertex-critical graphs. Such a characterization was known only in the case when H is connected. This solves a problem posed by Golovach *et al.* As a second result, we characterize all graphs H for which there are only finitely many H-free minimal obstructions for list 3-colorability.

The second result implies that when parameterized by the number of vertices that need to be removed to destroy all induced copies of a fixed connected graph H, the 3-colorability problem admits a polynomial kernel if and only if H is a path on at most six vertices.

1 Introduction

A k*-coloring* of a graph $G = (V, E)$ is a mapping $c : V \to \{1, \dots, k\}$ such that $c(u) \neq c(v)$ for all edges $uv \in E$. If a k-coloring exists, we say that G is k*-colorable*. The related decision problem – does a given input graph admit a k-coloring? – is called k*-colorability*; it is one of the most famous NP-complete problems. We say that G is k*-chromatic* if it is k-colorable but not $(k-1)$-colorable. In this paper we study the minimal obstructions for k-colorability: minimal subgraphs that hinder a graph from being k-colorable.

Recall that an *induced subgraph* is a subgraph obtained by vertex deletion, and it is called *proper* if at least one vertex was deleted. A graph is called $(k+1)$*-vertex-critical* if it is $(k+1)$-chromatic, but every induced proper subgraph is k-colorable. For example, the class of 3-vertex-critical graphs is the family of all odd cycles. In view of the NP-hardness of the k-colorability problem, there is little hope of giving a characterization of the $(k+1)$-vertex-critical graphs that is of use in algorithmic applications. The picture changes if one restricts the structure of the graphs under consideration, and the aim of this paper is to describe this phenomenon.

We use the following notation. If G and H are graphs such that G does not contain H as an induced subgraph we say that G is H*-free*. Moreover, we write $G + H$ for the graph that is the disjoint union of G and H. For all t, let P_t denote the path on t vertices.

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In an earlier work of ours, we proved the following theorem, solving a problem posed by Golovach *et al.* [13] and answering a question of Seymour [26].

Theorem 1 (Chudnovsky *et al.* [5])**.** *Let H be a connected graph. There are only finitely many 4-vertex-critical H-free graphs if and only if H is a subgraph of P_6.*

Let us stress the fact that H being a subgraph of P_6 includes the case of $H = P_6$. In view of Theorem 1, Golovach *et al.* [13] posed the problem of proving a similar characterization in the case when H is an arbitrary graph.

1.1 Our contribution

Our first main result solves this problem. We obtain the following dichotomy.

Theorem 2. *Let H be a graph. There are only finitely many H-free 4-vertex critical graphs if and only if H is a subgraph of P_6, $2P_3$, or $P_4 + kP_1$ for some $k \in \mathbb{N}$.*

We remark that the proof of Theorem 2 uses a different method than that of Theorem 1. Our new approach is more powerful in the sense that we can deal with both, connected and disconnected graphs H. The idea is to transfer the problem to the more general list setting defined below, and solve it there. This generality comes at a certain cost: we can only give a very rough upper bound on the maximum number of vertices an H-free 4-vertex-critical graph can have, assuming this is a finite number.

Our second main result is the equivalent of Theorem 2 in the list setting. To state it, we need the following notation.

Let G be a graph and let L be a mapping that maps each vertex of G to a subset of $\{1, \ldots, k\}$. We call L a *list system* for G, and each set $L(v)$ a *list*. We say that the pair (G, L) is *colorable* if there is a k-coloring c of G with $c(v) \in L(v)$ for each $v \in V(G)$. The *list k-colorability problem* is the following: given a pair (G, L) with $L(v) \subseteq \{1, \ldots, k\}$ for each $v \in V(G)$, decide whether (G, L) is colorable. Note that the list k-colorability problem generalizes both the k-colorability problem and the precoloring extension problem. In the k-colorability problem we have $|L(v)| = k$ for all $v \in V(G)$, while in the precoloring extension problem we have $|L(v)| \in \{1, k\}$ for all $v \in V(G)$.

We call a pair (G, L) with $L(v) \subseteq \{1, 2, 3\}$, $v \in V(G)$, a *minimal list-obstruction* if (G, L) is not colorable but for all induced proper subgraphs H of G the pair $\big(H, L|_{V(H)}\big)$ is colorable. Here and throughout the article, $L|_{V(H)}$ denotes the restriction of L to the domain $V(H)$. For convenience, we may also write $L|_H$.

Theorem 3. *Let H be a graph. There are only finitely many H-free minimal list-obstructions if and only if H is a subgraph of P_6 or $P_4 + kP_1$ for some $k \in \mathbb{N}$.*

Note that there are infinitely many $2P_3$-free minimal list-obstructions while there are only finitely many 4-vertex-critical $2P_3$-free graphs. Thus, Theorem 2 is not a special case of Theorem 3 . In the next section we describe an application of our result that also serves as a motivation for the concept of minimal list-obstructions.

1.2 An application: kernelizing 3-colorability

Let us now sketch an application of our results. It concerns the parameterized complexity of the 3-colorability problem and is based on work by Jansen and Kratsch [17].

We briefly review the relevant concepts from parameterized complexity. A decision problem parameterized by a problem-specific parameter r is called *fixed-parameter tractable* if there exists an algorithm that solves it in time $f(r) \cdot n^{O(1)}$, where n is the input size. One of the main tools to design such algorithms is the *kernelization* technique. A kernelization algorithm transforms in polynomial time an instance I of a given problem parameterized by r into an equivalent instance I' of the same problem parameterized by $r' \leq r$ such that the size of I' is bounded by $g(r)$ for some computable function g. The instance I' is called a *kernel* of size $g(r)$. The following result is well known.

Theorem 4. *A parameterized problem is fixed-parameter tractable if and only if it admits a kernel.*

Typically, even if a problem is known to admit a kernel, one wants to know whether the size of such a kernel can be bounded by a polynomial in the parameter. If this is the case, one speaks of a *polynomial kernel*. For more background on parameterized complexity the reader is referred to Downey and Fellows [8] and Flum and Grohe [10].

Let $\mathcal{G}$ be a hereditary graph class, that is, a graph class closed under deleting vertices. We denote by $\mathcal{G} + rv$ the class of graphs that can be turned into a member of $\mathcal{G}$ by the deletion of at most r vertices. Note that $\mathcal{G} + 0v = \mathcal{G}$ and that $\bigcup_{r \geq 0} \mathcal{G} + rv$ is the class of all graphs.

It is a natural question to ask whether the k-colorability problem is fixed parameter tractable in the graph class $\mathcal{G} + rv$ when r is the parameter. Obviously, a necessary condition is that the k-colorability problem is solvable in polynomial time on $\mathcal{G}$. Formally, the input of the k-colorability problem in the class $\mathcal{G} + rv$ is a graph G together with a vertex subset $X \subseteq V(G)$, called the *modulator*, such that $|X| \leq r$ and $G - X \in \mathcal{G}$. This assumption is made to avoid the potential computational difficulties the computation of a modulator might cause.

The following theorem shows that a finite list of minimal obstructions for list k-colorability in the class $\mathcal{G}$ implies the existence of a polynomial kernel for the k-colorability problem in the class $\mathcal{G} + rv$.

Theorem 5 (Jansen and Kratsch [17])**.** *Let $\mathcal{G}$ be a hereditary graph class such that every obstruction for list k-colorability in the class $\mathcal{G}$ has order at most s. Then the k-colorability problem in the class $\mathcal{G} + rv$ admits a polynomial kernel of order $O(r^{ks})$.*

Together with Theorem 3 and the construction given in Section 3.2 we obtain the following result.

Theorem 6. *Let H be a connected graph. The 3-colorability problem admits a polynomial kernel of order $O(r^{O(1)})$ on the class of graphs that can be made H-free by the removal of r vertices if and only if H is a subgraph of P_6. The only if part holds under the assumption that NP $\nsubseteq$ coNP/poly.*

We remark that Theorem 6 holds even if the modulator is not a part of the input. An easy greedy algorithm yields a 6-approximation of the problem of hitting all induced P_6 in a given graph, and thus one can compute a modulator of size at most $6r$ in polynomial time. This yields a polynomial time algorithm to compute a polynomial kernel of order $O(r^{O(1)})$.

1.3 Previous work

Let us now mention a few results in this line of research. It is known that the k-colorability problem is NP-hard on H-free graphs if H is any graph other than a subgraph of a path [16, 18, 19, 21]. This motivates the study of graph classes in which a path is forbidden as an induced subgraph. Regarding the computational complexity of the 3-colorability problem, the state of the art is the polynomial time algorithm to decide whether a P_7-free graph admits a 3-coloring [1]. The algorithm actually solves the harder list 3-colorability problem defined in the discussion above Theorem 3.

There are quite a few results regarding the number of critical H-free graphs. To describe these results, consider the following definition. If H is a graph, a $(k+1)$-*critical H-free graph* is a graph that is H-free, $(k+1)$-chromatic, and every H-free proper subgraph is k-colorable. Note that there are finitely many $(k+1)$-critical H-free graphs if and only if there are finitely many $(k+1)$-vertex-critical H-free graphs. For all t, we let C_t denote the cycle on t vertices.

Bruce *et al.* [3] proved that there are exactly six 4-critical P_5-free graphs. Later, Maffray and Morel [22], by characterizing the 4-vertex-critical P_5-free graphs, designed a linear time algorithm to decide 3-colorability of P_5-free graphs. Randerath et al. [25] have shown that the only 4-critical (P_6, C_3)-free graph is the Grötzsch graph. More recently, Hell and Huang [14] proved that there are four 4-critical (P_6, C_4)-free graphs. They also proved that there are only finitely many k-critical (P_6, C_4)-free graphs, for all k. As mentioned earlier, we proved Theorem 1 which says that there are only finitely many 4-critical P_6-free graphs, namely 24.

Recently [12], two of the authors of this paper developed an enumeration algorithm to automate the case distinctions performed in the proofs of the results mentioned above. Using this algorithm, it was shown that there are only finitely many 4-critical (P_7, C_k)-free graphs, for both $k = 4$ and $k = 5$. Since there is an infinite family of (P_7, C_6, C_7)-free graphs, only the case of (P_7, C_3)-free graphs remains open. It was also shown that there are only finitely many 4-critical (P_8, C_4)-free graphs.

For more details on this line of research we recommend the two excellent survey papers by Hell and Huang [15] and Golovach *et al.* [13].

Using the algorithm from [12] and adapting it to the list case, we were able to determine the exact number of P_6-free minimal list-obstructions with at most 9 vertices. It turns out that there are many such obstructions compared to the fact that there are only 80 P_6-free 4-vertex-critical graphs [5]. The counts can be found in Table 1 and our implementation of this algorithm can be downloaded from [11]. All obstructions up to 9 vertices can also be downloaded from the *House of Graphs* [2] at `http://hog.grinvin.org/Critical`. We remark that even though the algorithm of [12] was adapted to use in this paper, we also have a computer-free proof of all the results we need.

Vertices	1	2	3	4	5	6	7	8	9
Obstructions	1	1	4	43	117	1 806	34 721	196 231	1 208 483

Table 1: Counts of all P_6-free minimal list-obstructions with at most 9 vertices, up to swapping the lists and labels of the vertices. In total, there are 1 441 407 such list-obstructions.

1.4 Structure of the paper

In Section 2 we state the relevant definitions and the notation used in later sections. While most definitions are standard, we need to introduce a number of new concepts, mostly related to the various ways in which we update list systems.

In Section 3 we prove the necessity in the statement of Theorem 2 and Theorem 3, providing infinite families of H-free 4-vertex-critical graphs and minimal list-obstructions.

Section 4 deals with minimal list-obstructions in which every list has size at most 2. We introduce a concept, so-called *propagation paths*, that allows for quick and systematic proofs that there are only finitely many H-free minimal list-obstructions (for the right choices of H) with lists of size at most 2.

In Section 5 we prove that there are only finitely many P_6-free minimal list-obstructions. We split the proof into two steps, the first step being a proof of the fact that there are only finitely many P_6-free minimal list-obstructions where every list is of size at most 2. This step has both a computer-aided and a computer-free proof and uses the propagation path concept. The second step consists of reducing the general problem to the case solved in the first step. Here we rely on a structural analysis by hand, making use of a structure theorem for P_t-free graphs.

Using a similar approach, we prove in Section 6 that there are only finitely many $2P_3$-free 4-vertex-critical graphs. Moreover, we prove in Section 7 that there are only finitely many $P_4 + kP_1$-free minimal list obstructions.

In Section 9 we prove Theorem 6 building on the results from earlier sections and a method due to Jansen and Kratsch [17].

We close the paper by stating two problems for further research in Section 10.

2 Preliminaries

All graphs in this paper are finite and simple. Let G be a graph and let X be a subset of $V(G)$. We denote by $G[X]$ *the subgraph of G induced by X*, that is, the subgraph of G with vertex set X such that two vertices are adjacent in $G[X]$ if and only if they are adjacent in G. Moreover, we denote by $G \setminus X$ the graph $G[V(G) \setminus X]$. If $X = \{v\}$ for some $v \in V(G)$, we write $G \setminus v$ instead of $G \setminus \{v\}$. Let H be a graph. If G has no induced subgraph isomorphic to H, then we say that G is *H-free*. For a family $\mathcal{F}$ of graphs, we say that G is *$\mathcal{F}$-free* if G is F-free for every $F \in \mathcal{F}$. If G is not H-free, then G *contains* H. If $G[X]$ is isomorphic to H, then we say that X *is an H in G*. We write $G_1 + \ldots + G_k$ for the disjoint union of graphs $G_1, \ldots, G_k$.

For $n \geq 0$, we denote by P_{n+1} *the path with $n+1$ vertices and length n*, that is, the graph with distinct vertices $\{p_0, p_1, \ldots, p_n\}$ such that p_i is adjacent to p_j if and only if $|i-j| = 1$. For $n \geq 3$, we denote by C_n *the cycle of length n*, that is, the graph with distinct vertices $\{c_1, \ldots, c_n\}$ such that c_i is adjacent to c_j if and only if $|i-j| = 1$ or $|i-j| = n-1$. By convention, when explicitly describing a path or a cycle, we always list the vertices in order. Let G be a graph. When $G[\{p_0, p_1, \ldots, p_n\}]$ is the path P_{n+1}, we say that *p_0-p_1-…-p_n is a P_{n+1} in G*. Similarly, when $G[\{c_1, c_2, \ldots, c_n\}]$ is the cycle C_n, we say that *c_1-c_2-…-c_n-c_1 is a C_n in G*. We also refer to a cycle of length three as a *triangle*.

Let G be a graph. The *neighborhood* of a vertex $v \in V(G)$ is the set of all vertices adjacent to v, and is denoted by $N(v)$. For a vertex set S, we use $N(S)$ to denote $\bigcup_{v \in S} N(v)$. A *clique* in G is a set of vertices which are all pairwise adjacent, and a *stable set* is a set of vertices which are all pairwise non-adjacent. A *Hamiltonian path* is a path that contains all vertices of G.

A *partition* of a set S is a collection of disjoint subsets of S whose union is S. Let A and B be disjoint subsets of $V(G)$. For a vertex $b \in V(G) \setminus A$, we say that *b is complete to A* if b is adjacent to every vertex of A, and that *b is anticomplete to A* if b is non-adjacent to every vertex of A. If every vertex of A is complete to B, we say *A is complete to B*, and if every vertex of A is anticomplete to B, we say that *A is anticomplete to B*. If $b \in V(G) \setminus A$ is neither complete nor anticomplete to A, we say that *b is mixed on A*. We say G is *connected* if $V(G)$ cannot be partitioned into two disjoint non-empty sets anticomplete to each other. The *complement* $\overline{G}$ of G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G. If $\overline{G}$ is connected we say that G is *anticonnected*. For $X \subseteq V(G)$, we say that X is *connected* if $G[X]$ is connected, and that X is anticonnected if $G[X]$ is anticonnected. A *component* of $X \subseteq V(G)$ is a maximal connected subset of X, and an *anticomponent* of X is a maximal anticonnected subset of X. We write *component of G* to mean a component of $V(G)$. A subset D of $V(G)$ is called a *dominating set* if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D; in this case we also say that $G[D]$ is a *dominating subgraph* of G. We say that G is a *bipartite graph* with *bipartition* (A, B) if $V(G)$ can be partitioned into two disjoint sets, A and B, such that both of them are stable sets.

A *k-coloring* of a graph G is a mapping $c : V(G) \to \{1, \ldots, k\}$ such that if $x, y \in V(G)$ are adjacent, then $c(x) \neq c(y)$. If a k-coloring exists for a graph G, we say that G is *k-colorable*. A *list system* L of a graph G is a mapping which assigns each vertex $v \in V(G)$ a finite subset of $\mathbb{N}$, denoted by $L(v)$. Each $L(v)$, $v \in V(G)$, we call a *list*. A *subsystem* of a list system L of G is a list system L' of G such that $L'(v) \subseteq L(v)$ for all $v \in V(G)$. We say a list system L of the graph G has *order k* if $L(v) \subseteq \{1, \ldots, k\}$ for all $v \in V(G)$. In this article, we will only consider list systems of order 3. Notationally, we write (G, L) to represent a graph G and a list system L of G. We say that c, a coloring of G, is an *L-coloring* of G, or a *coloring of* (G, L) provided $c(v) \in L(v)$ for all $v \in V(G)$. We say that (G, L) is *colorable*, if there exists a coloring of (G, L).

A *partial coloring* is a mapping $c : U \to \mathbb{N}$ such that $c(u) \in L(u)$ for all $u \in U$, where U is a subset of $V(G)$. Note that here we allow for edges uv of $G[U]$ with $c(u) = c(v)$. If there is no such edge, we call c *proper*.

Let G be a graph and let L be a list system of order 3 for G. We say that (G, L) is a *list-obstruction* if (G, L) is not colorable. As stated earlier, we call (G, L) a *minimal list-obstruction* if, moreover, $(G \setminus x, L|_{G \setminus x})$ is colorable for every vertex $x \in G$.

Let (G, L) be a list-obstruction. We say a vertex $v \in V(G)$ is *critical* if $G \setminus v$ is L-colorable and *non-critical* otherwise. If we repeatedly delete non-critical vertices of G until the new graph, G' say, with list system $L|_{G'}$ is a list-obstruction, we say that $(G', L|_{G'})$ is *induced* by (G, L).

Let u, v be two vertices of a list-obstruction (G, L). We say that u *dominates* v if $L(u) \subseteq L(v)$ and $N(v) \subseteq N(u)$. It is easy to see that if there are such vertices u and v in G, then (G, L) cannot be a minimal list-obstruction. We frequently use this observation without further reference.

2.1 Updating lists

Throughout the proof, we make use of distinct updating procedures to reduce the sizes of the lists. Let G be a graph and let L be a list system for G.

Let $v, w \in V(G)$ be adjacent. An *update* of the list of the vertex v *from* w means we delete an entry from the list of v that appears as the unique entry of the list of w. Clearly, such an update does not change the colorability of the graph.

If $P = v_1\text{-}\ldots\text{-}v_k$ is a path, then *updating from* v_1 *along* P means that we assign a color to v_1 and then update v_2 from v_1 if possible. Then we update v_3 from v_2, and so on. When v_k is updated from v_{k-1}, we stop the updating.

Let $X \subseteq V(G)$ such that $|L(x)| = 1$ for all $x \in X$. For a subset $Y \subseteq V(G) \setminus X$, we say that we *update the lists of the vertices in* Y *with respect to* X if we update each $y \in Y$ from each $x \in X$. We say that we *update the lists with respect to* X if $Y = V(G) \setminus X$. Let $X_0 = X$ and $L_0 = L$. For $i \geq 1$ define X_i and L_i as follows. L_i is the list system obtained from L_{i-1} by updating with respect to X_{i-1}. $X_i = X_{i-1} \cup \{v \in V(G) \setminus X_{i-1}$ such that $|L_i(v)| = 1$ and $|L_{i-1}(v)| > 1\}$. We say that L_i is obtained from L by *updating with respect to* X i *times*. If $X = \{w\}$ we say that L_i is obtained by *updating with respect to* w i *times*. If for some i, $W_i = W_{i-1}$ and $L_i = L_{i-1}$, we say that L_i was obtained from L by *updating exhaustively with respect to* X *(or* w*)*. For simplicity, if X is a subgraph of G, by updating with respect to X we mean updating with respect to $V(X)$.

2.2 Reducing obstructions

In this section we prove three lemmas which help us reduce the size of the obstructions. These lemmas will be used in the proofs of Sections 5.2 and 6.2.

Let (G, L) be a P_6-free list-obstruction and let R be an induced subgraph of G. We will now define a set of subsystems of the list system L with the property that every coloring of $(R, L|_R)$ is a coloring of R in one of the restricted list systems. Let $\mathcal{L}$ be a set of subsystems of L satisfying the following assertions.

1. For each $L' \in \mathcal{L}$, $L'(v) = L(v)$ for $v \in G \setminus R$ and $L'(v) \subseteq L(v)$ for $v \in R$.
2. For every $L|_R$-coloring c of R there exists a list system $L' \in \mathcal{L}$ such that c is also an $L'|_R$-coloring of R.

We call $\mathcal{L}$ a *refinement of L with respect to R*. For each list systems $L' \in \mathcal{L}$ it is clear that (G, L') is again a list-obstruction, though not necessarily a minimal one, even if (G, L) is minimal.

Lemma 7. *Assume that (G, L) is a minimal list-obstruction. Let R be an induced subgraph of G, and let $\mathcal{L} = \{L_1, L_2, \dots, L_m\}$ be a refinement of L with respect to R. Moreover, assume that for every $L' \in \mathcal{L}$ some minimal list-obstruction induced by (G, L') contains at most k vertices apart from the vertices in R. Then G has at most $|R| + km$ vertices.*

Proof. Let $(G_i, L_i|_{G_i})$ be a minimal list-obstruction induced by (G, L_i) such that $|V(G_i) \setminus V(R)| \leq k$, $i = 1, \dots, m$. Suppose for a contradiction that there exists a vertex v in G such that v is contained in none of the G_i, $i = 1, \dots, m$. By the minimality of (G, L), $G \setminus \{v\}$ is $L|_{G\setminus\{v\}}$-colorable. Let c be an $L|_{G\setminus\{v\}}$-coloring of $G \setminus \{v\}$. We may assume that $c(r) \in L_1(r)$ for every $r \in V(R) \cap V(G_1)$. Then c is a coloring of $(G_1, L_1|_{G_1})$, which is a contradiction. Hence,

$$|G| \leq |\bigcup_{i=1}^{m} V(G_i)| \leq |R| + |\bigcup_{i=1}^{m} V(G_i \setminus R)| \leq |R| + km,$$

as desired. □

Next we prove a lemma which allows us to update three times with respect to a set of vertices with lists of size 1.

Lemma 8. *Let (G, L) be a list-obstruction, and let $X \subseteq V(G)$ be a vertex subset such that $|L(x)| = 1$ for every $x \in X$. Let L' be the list obtained by updating with respect to X three times. Let $(G', L'|_{G'})$ be a minimal list-obstruction induced by (G, L'). Then there exists a minimal list-obstruction induced by (G, L), say $(G'', L|_{G''})$, with $|V(G'')| \leq 36|V(G')|$.*

Proof. Let $Y = X_1$ and $Z = X_2$, as in the definition of updating i times. We choose sets R, S, and T as follows.

- For every $v \in V(G') \setminus (X \cup Y \cup Z)$, define $R(v)$ to be a minimum subset of $(X \cup Y \cup Z) \cap N(v)$ such that $\bigcup_{s \in R(v)} L'(s) = L(v) \setminus L'(v)$, and let $R = \bigcup_{v \in V(G') \setminus (X \cup Y \cup Z)} R(v)$.
- For every $v \in (V(G') \cup R) \cap Z$, define $S(v)$ to be a minimum subset of $(X \cup Y) \cap N(v)$ such that $\bigcup_{s \in S(v)} L'(s) = L(v) \setminus L'(v)$, and let $S = \bigcup_{v \in (V(G') \cup R) \cap Z} S(v)$.
- For every $v \in (V(G') \cup R \cup S) \cap Y$, define $T(v)$ to be a minimum subset of $X \cap N(v)$ such that $\bigcup_{s \in T(v)} L'(s) = L(v) \setminus L'(v)$, and let $T = \bigcup_{v \in (V(G') \cup R \cup S) \cap Y} T(v)$.

Clearly, $|R(v)| \leq 3$ for every $v \in V(G') \setminus (X \cup Y \cup Z)$, $|S(v)| \leq 2$ for every $v \in (V(G') \cup R) \cap Z$, and $|T(v)| \leq 2$ for every $v \in (V(G') \cup R \cup S) \cap Y$. Let $P = R \cup S \cup T \cup V(G')$, and observe that $|P| \leq (1 + 3 + 8 + 24)|V(G')| = 36|V(G')|$. It remains to prove that $(G[P], L|_{G[P]})$ is not colorable. Suppose there exists a coloring c of $(G[P], L|_{G[P]})$. Then c is not a coloring of $(G', L'|_{G'})$, and since $V(G') \subseteq P$, it follows that there exists $w \in V(G')$ such that $c(w) \notin L'(w)$. Therefore $c(w) \in L(w) \setminus L'(w)$.

We discuss the case when $v \in V(G') \setminus (X \cup Y \cup Z)$, as the cases of $v \in (V(G') \cup R) \cap Z$ and $v \in (V(G') \cup R \cup S) \cap Y$ are similar. We can choose $m \in R(w)$ such that $L'(m) = \{c(w)\}$ and one of the following holds.

- $m \in X$, and thus $L(m) = L'(m) = \{c(w)\}$.
- $m \in Y$, and thus for any $i \in L(m) \setminus L'(m)$ there exists $n_i \in T(m)$ such that $L(n_i) = \{i\}$.
- $m \in Z$, and thus for any $i \in L(m) \setminus L'(m)$ there exists $n_i \in S(m)$ such that either $L(n_i) = \{i\}$ or, for any $j \in L(n_i) \setminus \{i\}$, there exists $l_j \in T(m)$ with $L(l_j) = \{j\}$.

In all cases it follows that $c(m) = c(w)$, in contradiction to the fact that m and w are adjacent. This completes the proof. □

Let A be a subset of $V(G)$ and L be a list system; let c be a $L|_{G[A]}$-coloring of $G[A]$, and let L_c be the list system obtained by setting $L_c(v) = \{c(v)\}$ for every $v \in A$ and updating with respect to A three times; we say that L_c is *obtained from L by precoloring A (with c) and updating three times*. If for every c, $|L_c(v)| \leq 2$ for every $v \in V(G)$, we call A a *semi-dominating set* of (G, L). If $L(v) = \{1, 2, 3\}$ for every $v \in G$ and A is a semi-dominating set of (G, L), we say that A is a *semi-dominating set* of G. Note that a dominating set is always a semi-dominating set. Last we prove a lemma for the case when G has a bounded size semi-dominating set.

Lemma 9. *Let (G, L) be a minimal list-obstruction, and assume that G has a semi-dominating set, A, of size at most t. Assume also that if (G', L') is a minimal obstruction where G' is an induced subgraph of G, and L' is a sublist of L with $|L'(v)| \leq 2$ for every v, then $|V(G')| \leq m$. Then $|V(G)| \leq 36 \cdot 3^t \cdot m + t$.*

Proof. Consider all possible $L|_A$-colorings $c_1, \ldots, c_s$ of A; then $s \leq 3^t$. For each i, let L_i be the list system obtained by updating with respect to A three times. Then $|L_i(v)| \leq 2$ for every $v \in V(G)$ and for every $i \in \{1, \ldots, s\}$. Now Lemma 7 together with Lemma 8 imply that $|V(G)| \leq 36 \cdot 3^t \cdot m + t$. This completes the proof. □

3 Necessity

The aim of this section is to prove the following two statements.

Lemma 10. *There are infinitely many H-free 4-vertex-critical graphs if H is a claw, a cycle, or $2P_2 + P_1$.*

Here, a *claw* is the graph consisting of a central vertex plus three pendant vertices attached to it. In the list-case, the following variant of this statement holds.

Lemma 11. *There are infinitely many H-free minimal list-obstructions if H is a claw, a cycle, $2P_2 + P_1$, or $2P_3$.*

We remark that Lemma 10 implies the following. Whenever H is a graph containing a claw, a cycle, or $2P_2 + P_1$ as an induced subgraph, there are infinitely many H-free 4-vertex-critical graphs. A similar statement is true with respect to Lemma 11 and minimal list-obstructions.

3.1 Proof of Lemma 10

Recall that there are infinitely many 4-vertex-critical claw-free graphs. For example, this follows from the existence of 4-regular bipartite graphs of arbitrarily large girth (cf. [20] for an explicit construction of these) whose line graphs are necessarily 4-chromatic. Moreover, there are 4-chromatic graphs of arbitrarily large girth, which follows from a classical result of Erdős [9]. This, in turn, implies that there exist 4-vertex-critical graphs of arbitrary large girth. Putting these two remarks together, we see that if H is the claw or a cycle, then there are infinitely many 4-vertex-critical graphs.

We now recall a construction due to Pokrovskiy [23] which gives an infinite family of 4-vertex-critical P_7-free graphs. It is presented in more detail in our earlier work [5].

For each $r \geq 1$, let G_r be the graph defined on the vertex set $v_0, \ldots, v_{3r}$ with edges as follows. For all $i \in \{0, 1, \ldots, 3r\}$ and $j \in \{0, 1, \ldots, r-1\}$, the vertex v_i is adjacent to v_{i-1}, v_{i+1}, and v_{i+3j+2}. Here, we consider the indices to be taken modulo $3r + 1$. The graph G_5 is shown in Figure 1.

Up to permuting the colors, there is exacly one 3-coloring of $G_r \setminus v_0$. Indeed, we may assume that v_i receives color i, for $i = 1, 2, 3$, since $\{v_1, v_2, v_3\}$ forms a triangle in G_r. Similarly, v_4 receives color 1, v_5 receives color 2 and so on. Finally, v_{3r} receives color 3. It follows that G_r is not 3-colorable, since v_0 is adjacent to all of v_1, v_2, v_{3r}.

As the choice of v_0 was arbitrary, we know that G_r is 4-vertex-critical. The graph G_r is $2P_2 + P_1$-free which can be seen as follows.

Claim 1. *For all r the graph G_r is $2P_2 + P_1$-free.*

Proof. Suppose there is some r such that G_r is not $2P_2 + P_1$-free. Let $v_{i_1}, \ldots, v_{i_5}$ be such that $G_r[\{v_{i_1}, \ldots, v_{i_5}\}]$ is a $2P_2 + P_1$. Since G_r is vertex-transitive, we may assume that $i_1 = 1$ and $N(v_{i_1}) \cap \{v_{i_2}, \ldots, v_{i_5}\} = \emptyset$. In particular, $i_2, \ldots, i_5 \neq 0$.

Consider $G_r \setminus v_0$ to be colored by the coloring c proposed above, where each v_i receives the color $i \mod 3$. Due to the definition of G_r, v_{i_1} is adjacent to every vertex of color 3, and thus $c(v_j) \neq 3$ for all $j \in \{i_2, \ldots, i_5\}$.

We may assume that $c(v_{i_2}) = c(v_{i_4}) = 1$, $c(v_{i_3}) = c(v_{i_5}) = 2$, and both $v_{i_2}v_{i_3}$ and $v_{i_4}v_{i_5}$ are edges of $E(G_r)$. For symmetry, we may further assume that $i_2 < i_4$. Due to the definition of G_r, v_{i_2} and v_{i_4} are adjacent to every vertex of color 2 with a smaller index, and thus $i_4 < i_3$. But now $i_2 < i_4 < i_3$, a contradiction to the fact that $v_{i_2}v_{i_3} \in E(G_r)$. This completes the proof. □

Consequently, there are infinitely many $2P_2 + P_1$-free 4-vertex-critical graphs, as desired.

3.2 Proof of Lemma 11

According to Lemma 10, it remains to prove that there are infinitely many $2P_3$-free minimal list-obstructions.

For all $r \in \mathbb{N}$, let H_r be the graph defined as follows. The vertex set of H_r is $V(H_r) = \{v_i : 1 \leq i \leq 3r - 1\}$. There is an edge from v_1 to v_2, from v_2 to v_3 and so on. Thus, $P := v_1\text{-}v_2\text{-}\ldots\text{-}v_{3r-1}$ is a path. Moreover, there is an edge between a vertex v_i and a vertex v_j if $i \leq j - 2$, $i \equiv 2 \mod 3$, and $j \equiv 1 \mod 3$. There are no further edges. The graph H_5 is shown in Figure 2.

The list system L is defined by $L(v_1) = L(v_{3r-1}) = \{1\}$ and, assuming $2 \leq i \leq 3r - 2$,

$$L(v_i) = \begin{cases} \{2, 3\}, & \text{if } i \equiv 0 \mod 3 \\ \{1, 3\}, & \text{if } i \equiv 1 \mod 3 \\ \{1, 2\}, & \text{if } i \equiv 2 \mod 3 \end{cases}.$$

Next we show that the above construction has the desired properties.

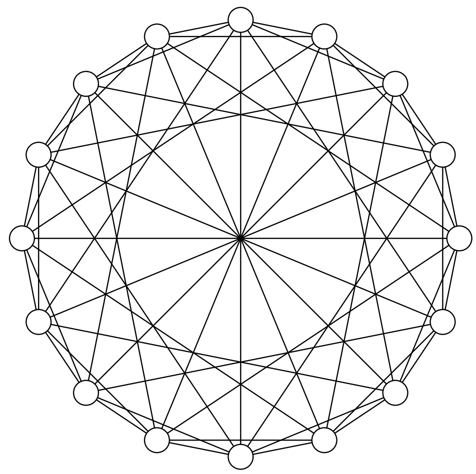

Figure 1: A circular drawing of G_5

Claim 2. *The pair (H_r, L) is a minimal $2P_3$-free list-obstruction for all r.*

Proof. Let us first show that, for any r, H_r is not colorable. Consider the partial coloring c that assigns color 1 to v_1. Since $L(v_2) = \{1, 2\}$, the coloring can be updated from v_1 to v_2 by putting $c(v_2) = 2$. Now we can update the coloring from v_2 to v_3 by putting $c(v_3) = 3$. Like this we update the coloring along P until v_{3r-2} is colored. However, we have to put $c(v_{3r-2}) = 1$, in contradiction to the fact that $L(v_{3r-1}) = \{1\}$. Thus, H_r is not colorable.

Next we verify that (H_r, L) is a minimal list-obstruction. If we delete v_1 or v_{3r-1}, the graph becomes colorable. So let us delete a vertex v_i with $2 \leq i \leq 3r-2$. We can color $(H_r \setminus v_i, L_{H_r \setminus v_i})$ as follows. Give color 1 to v_1 and update along P up to v_{i-1}. Moreover, give color 1 to v_{3r-1} and update along P backwards up to v_{i+1}. Call this coloring c.

To check that c is indeed a coloring, we may focus on the non-path edges for obvious reasons. Pick an edge between a vertex v_j and a vertex v_k with $j \leq k-2$, if any. By definition, $j \equiv 2 \mod 3$ and $k \equiv 1 \mod 3$. If $j < i < k$, $c(v_j) = 2$ and $c(v_k) = 3$. Moreover, if $j < k < i$, $c(v_j) = 2$ and $c(v_k) = 1$. Finally, if $i < j < k$, $c(v_j) = 1$ and $c(v_k) = 3$. So, c is indeed a coloring of $H_r \setminus v_i$ and it remains to prove that H_r is $2P_3$-free.

Suppose this is false, and let r be minimum such that H_r contains an induced $2P_3$. Let F be a copy of such a $2P_3$ in H_r. It is clear that $r \geq 2$. Note that $H_r \setminus N(v_2)$ is the disjoint union of complete graphs of order 1 and 2, and so $v_2 \notin V(F)$. Since $N(v_1) = \{v_2\}$, we know that $v_1 \notin V(F)$. Moreover, as $F \setminus (N(v_5) \cup \{v_1, v_2\})$ is the disjoint union of complete graphs of order 1 and 2, we deduce that $v_5 \notin V(F)$. But $F' := F \setminus \{v_1, v_2, v_3\}$ is isomorphic to H_{r-1}, and thus the choice of r implies that F' is $2P_3$-free. Consequently, $v_3 \in V(F)$. Since $N(v_3) = \{v_2, v_4\}$ and $v_2 \notin V(F)$, we know that $v_4 \in V(F)$. Finally, the fact that $N(v_4) = \{v_2, v_3, v_5\}$ implies that v_3 and v_4 both have degree one in F, and they are adjacent, a contradiction. □

4 Obstructions with lists of size at most two

The aim of this section is to prove an upper bound on the order of the H-free minimal list-obstructions in which every list has at most two entries. Let us stress the fact that we restrict

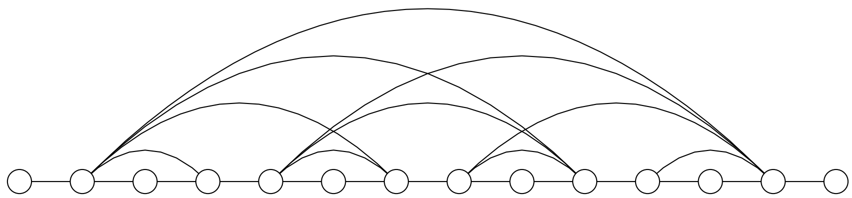

Figure 2: A drawing of H_5. The vertices v_1 to v_{14} are shown from left to right.

our attention to lists of order 3. Before we state our lemma, we need to introduce the following technical definition.

Let (G, L) be a minimal list-obstruction such that $|L(v)| \leq 2$ for all $v \in V(G)$. Let $P = v_1$-v_2-...-v_k be a path in G, not necessarily induced. Assume that $|L(v_1)| \geq 1$ and $|L(v_i)| = 2$ for all $i \in \{2, \ldots, k\}$. Moreover, assume that there is a color $\alpha \in L(v_1)$ such that if we give color α to v_1 and update along P, we obtain a coloring c of P. Please note that c may not be a coloring of the graph $G[V(P)]$. For $i \in \{2, \ldots, k\}$ with $L(v_i) = \{\beta\gamma\}$ and $c(v_i) = \beta$ we say that $S(v_i) := \beta\gamma$ is the *shape* of v_i. Assume that every edge v_iv_j with $3 \leq i < j \leq n$ and $i \leq j - 2$ is such that

$$S(v_i) = \alpha\beta \text{ and } S(v_j) = \beta\gamma, \tag{1}$$

where $\{1, 2, 3\} = \{\alpha, \beta, \gamma\}$. Then we call P a *propagation path* of G and say that P *starts with color* α. As we prove later, (1) implies that the updating process from v_1 along P to v_k cannot be shortcut via any edge v_iv_j with $3 \leq i < j \leq k$ and $i \leq j - 2$.

The next lemma shows that, when bounding the order of our list-obstructions, we may concentrate on upper bounds on the size of propagation paths.

Lemma 12. *Let (G, L) be a minimal list-obstruction. Assume that all propagation paths in G have at most λ vertices for some $\lambda \geq 20$. Then G has order at most $4\lambda + 4$.*

In the next section we prove the above lemma.

We first show that each minimal list-obstruction in which every list contains at most two colors is the (not necessarily disjoint) union of two paths starting at a common vertex, where giving the common vertex either of its two colors and then updating along one of the two paths yields an improper coloring.

Next we prove a number of properties that both paths must satisfy, basically showing that certain types of edges cannot appear between vertices with particular lists. Exploiting these properties, we show that long propagation paths must appear. This proves the lemma.

4.1 Proof of Lemma 12

Let $(G = (V, E), L)$ be a minimal list-obstruction such that $|L(v)| \leq 2$ for all $v \in V$. If there is a vertex with an empty list, then this is the only vertex of G and we are done. So, we may assume that every vertex of G has a non-empty list. Let $V_1 = \{v \in V : |L(v)| = 1\}$ and $V_2 = \{v \in V : |L(v)| = 2\}$.

Claim 3. *Let $x \in V$ and $\alpha \in L(x)$ be arbitrary. Assume we give color α to x and update exhaustively in the graph $(G[V_2 \cup \{x\}], L|_{V_2 \cup \{x\}})$. Let c be the obtained partial coloring. For each $y \in V_1$ that did not recieve a color so far, let $c(y)$ be the unique color in $L(y)$. Then there is an edge uv such that $c(u) = c(v)$.*

Proof. Let us give color α to x and update exhaustively, but only considering vertices and edges in the graph $(G[V_2 \cup \{x\}], L|_{V_2\cup\{x\}})$. We denote this partial coloring by c. For each $y \in V_1$ that did not recieve a color so far, let $c(y)$ be the unique color in $L(y)$. For a contradiction, suppose that this partial coloring c is proper.

Since G is an obstruction, c is not a coloring of G, meaning there are still vertices with two colors left on their list. We denote the set of these vertices by U. By minimality of G, we know that both graphs $(G', L') := (G \setminus U, L|_{V\setminus U})$ and $(G'', L'') := (G[U \cup V_1] \setminus x, L|_{(U\cup V_1)\setminus\{x\}})$ are colorable and have at least one vertex.

Let c' be the coloring of G' such that $c'(u) = c(u)$ for all $u \in V(G')$, and let c'' be a coloring of G''. It is clear that c' and c'' agree on the vertices in $V(G') \cap V(G'') = V_1 \setminus \{x\}$. Moreover, if $v \in (V_2 \setminus U) \cup \{x\}$ and $u \in U$ such that $uv \in E(G)$, then $c(v) \notin L(u)$. Since $c'(v) = c(v)$ for every $v \in V(G')$, we deduce that $c'(v) \neq c''(u)$ for every $uv \in E(G)$ with $u \in U$ and $v \in (V_2 \setminus U) \cup \{x\}$. Consequently, we found a coloring of (G, L), a contradiction. Thus every vertex of V_2 receives a color in the partial coloring c, a contradiction. □

Claim 4. *It holds that* $|V_1| \leq 2$.

Proof. Suppose that $|V_1| \geq 3$ and let $x \in V_1$ and $\alpha \in L(x)$. Let us give color α to x and update exhaustively, but only considering vertices and edges in the graph $(G[V_2 \cup \{x\}], L|_{V_2\cup\{x\}})$. We denote this partial coloring by c.

Since G is minimal, there is no edge uv with $u, v \in V_2 \cup \{x\}$ and $c(u) = c(v)$. Since $(G|(V_1 \setminus \{x\}), L|_{V_1\setminus\{x\}})$ is colorable by the minimality of F, and since $|L(v)| = 1$ for every $v \in V_1 \setminus \{x\}$, it follows that there is an edge uv with $u \in V_2 \cup \{x\}$ and $v \notin V_2 \cup \{x\}$ such that $L(v) = \{c(u)\}$. It follows that $(G|(V_2\cup\{x, v\}), L|_{V_2\cup\{x,v\}})$ is not colorable, and so by the minimality of G, $V_1 = \{x, v\}$, as required. This proves the first claim. □

Next we prove that, loosely speaking, G is the union of at most two paths, starting in a common vertex, along each of which updating yields a contradiction. Depending on the cardinality of V_1, we arrive at three different situations which are described by the following three claims. Recall from Claim 4 that $|V_1| \leq 2$.

Claim 5. *Assume that* $|V_1| = 0$*, and pick* $x \in V$ *arbitrarily. Let us say that* $L(x) = \{1, 2\}$*. For* $\alpha = 1, 2$ *there is a path* $P^\alpha = v_1^\alpha\text{-}\ldots\text{-}v_{k_\alpha}^\alpha$*, not necessarily induced, with the following properties.*

(a) If we give color α *to* x *and update along* P^α*, then all vertices of* P^α *will be colored.*

(b) Assume that v_i^α *gets colored in color* γ_i*,* $i = 1, \ldots, k_\alpha$*. Then there is an edge of the form* $v_i^\alpha v_j^\alpha$ *with* $\gamma_i = \gamma_j$*.*

(c) $V = V(P^1) \cup V(P^2)$*.*

Proof. We give color α to x and update exhaustively from x. According to Claim 3, after some round of updating an edge appears whose end vertices receive the same color. We then stop the updating procedure. During the whole updating procedure we record an auxiliary digraph $D = (W, A)$ as follows. Initially, $W = \{x\}$ and $A = \emptyset$. Whenever we update a vertex u from a vertex v, we add the vertex u to W and the edge (v, u) to A. This gives a directed tree whose root is x.

We can find directed paths R and S in T both starting in x and ending in vertices y and z, say, such that y and z are adjacent in G and they receive the same color during the updating procedure. We may assume that $R = u_1\text{-}\ldots\text{-}u_k\text{-}v_1\text{-}\ldots\text{-}v_r$ and $S = u_1\text{-}\ldots\text{-}u_k\text{-}w_1\text{-}\ldots\text{-}w_s$, where R and S share only the vertices $u_1, \ldots, u_k$. For each vertex $v \in V(R) \cup V(S)$, let $c(v)$ be the color received by v in the updating procedure. Moreover, let $c'(v)$ be the unique color in $L(v) \setminus \{c(v)\}$. Observe that, setting $w_0 = v_0 = u_k$, we have that $c'(w_i) = c(w_{i-1})$ for every $i \in \{1, \ldots, s\}$, and $c'(v_i) = c(v_{i-1})$ for every $i \in \{1, \ldots, k\}$.

Consider the following, different updating with respect to x. We again give color α to x, and then update along R. Now we update w_s from v_r, thus giving it color $c'(w_s)$. This, in turn, means we can update w_{s-1} from w_s, giving it color $c'(w_{s-1})$, and so on. Finally, when we update w_1 and it recieves color $c'(w_1)$, an edge appears whose end vertices are colored in the same color. Indeed, $u_k w_1$ is such an edge since $c(u_k) = c'(w_1)$. Summing up, the path

$$P^\alpha = u_1\text{-}\ldots\text{-}u_k\text{-}v_1\text{-}\ldots\text{-}v_r\text{-}w_s\text{-}w_{s-1}\text{-}\ldots\text{-}w_1$$

starts in x and, when we give x the color α and update along P^α, we obtain an improper partial coloring. As $\alpha \in \{1, 2\}$ was arbitrary, the assertions (a) and (b) follow.

To see (c), just note that the graph $G[V(P_1) \cup V(P_2)]$ is an obstruction: giving either color of $L(x)$ to x and updating exhaustively yields a monochromatic edge. By the minimality of G, $G = G[V(P_1) \cup V(P_2)]$ and so (c) holds. □

Claim 6. *Assume that $|V_1| = 1$, say $V_1 = \{x\}$ with $L(x) = \{\alpha\}$. Then the following holds:*

(a) there is a Hamiltonian path $P = v_1\text{-}\ldots\text{-}v_k$ of G, not necessarily induced, with $x = v_1$;

(b) updating from $x = v_1$ along P assigns a color γ_i to v_i, $i = 1, \ldots, k$; and

(c) there is an edge of the form $v_i v_j$ with $\gamma_i = \gamma_j$.

Proof. We assign color α to x and update exhaustively from x. Let c be the obtained partial coloring. According to Claim 3, there is an edge uv of G with $c(u) = c(v)$. Since (G, L) is a minimal obstruction, every vertex of G recieved a color in the updating process: otherwise, we could simply remove such a vertex and still have an obstruction.

Repeating the argument from the proof of Claim 5, we obtain a path P that starts in x and, when we give x color α and update along P, we obtain an improper partial coloring. This proves (b) and (c). Due to the minimality of (G, L), the path P is Hamiltonian, which proves (a). □

Claim 7. *Assume $|V_1| = 2$, say $V_1 = \{x, y\}$ with $L(x) = \{\alpha\}$ and $L(y) = \{\beta\}$. Then the following holds:*

(a) there is a Hamiltonian path $P = v_1\text{-}\ldots\text{-}v_k$ of G with $x = v_1$ and $y = v_k$; and

(b) updating from v_1 along P assigns the color β to v_{k-1}.

Proof. We color x with color α and update exhaustively from x, but only considering vertices and edges of the graph $G \setminus y$. Let c be the obtained partial coloring. By minimality, c is proper. According to Claim 3, there is a neighbor u of y in G with $c(u) = \beta$.

Like in the proof of Claim 5 and Claim 6, we see that there is a path P from x to y whose last edge is uy such that giving color α to x and then updating along P implies that u is colored with color β, which implies (b). Due to the minimality of (G, L), the path P is Hamiltonian, and thus (a) holds. □

We can now prove our main lemma.

Proof of Lemma 12. Suppose first that $|V_1| = 0$, then Claim 5 applies and we obtain x, P^1 and P^2 as in the statement of the claim. We may assume that, among all possible choices of x, P^1 and

P^2, the value $\max\{|V(P^1)|, |V(P^2)|\}$ is minimum and, subject to this, $\min\{|V(P^1)|, |V(P^2)|\}$ is minimum.

Let us say that $P^1 = v_1\text{-}v_2\text{-}\dots\text{-}v_s$, where $v_1 = x$. Consider v_1 to be colored in color 1, and update along P^1, but only up to v_{s-1}. Due to the choice of P^1 and P^2 being of minimum length, the coloring so far is proper. Now when we update from v_{s-1} to v_s, two adjacent vertices recieve the same color. Let the partial coloring obtained so far be denoted c. Let X be the set of neighbors w of v_s on P^1 with $c(w) = c(v_s)$, and let r be minimum such that $v_r \in X$.

We claim that $s - r \leq \lambda$. To see this, let $c'(v_j)$ be the unique color in $L(v_j) \setminus \{c(v_j)\}$, for all $j = 1, \dots, s$. We claim that the following assertions hold.

(a) $c(v_j) = c'(v_{j+1})$ for all $j = r, \dots, s-1$.

(b) For every edge v_iv_j with $r \leq i, j \leq s-1$ it holds that $c(v_i) \neq c(v_j)$.

(c) For every edge v_iv_j with $r \leq i, j \leq s$ and $j - i \geq 2$ it holds that $c(v_i) \neq c'(v_j)$.

(d) For every edge v_iv_j with $r + 2 \leq i, j \leq s$ it holds that $c'(v_i) \neq c'(v_j)$.

Assertion (a) follows from the fact that P obeys the assertions of Claim 5. For (b), note that the choice of P_1 to be of minimum length implies that until we updated v_s, the partial coloring is proper.

To see (c), suppose there is an edge v_iv_j with $r \leq i, j \leq s$ and $j - i \geq 2$ such that $c(v_i) = c'(v_j)$. Then the path P^1 can be shortened to the path $v_1\text{-}\dots\text{-}v_i\text{-}v_j\text{-}\dots\text{-}v_s$, which is a contradiction.

Now we turn to (d), and consider the following coloring. We color P^1 as before up to v_r. Now we update from v_r to v_s, giving color $c'(v_s)$ to v_5. Then we color v_{s-1} with color $c'(v_{s-1})$, then v_{s-2} with color $c'(v_{s-2})$, and so on, until we reach v_{r+1}. But $c'(v_{r+1}) = c(v_r)$ due to (a), which means that the path $Q^1 = v_1\text{-}v_2\text{-}\dots\text{-}v_r\text{-}v_s\text{-}v_{s-1}\text{-}v_{s-2}\text{-}\dots\text{-}v_{r+1}$ is a choice equivalent to P^1. In particular, due to the choice of P^1 and P^2, the constructed coloring of Q^1 is proper if we leave out v_{r+1}. Hence, there is no edge v_iv_j with $r + 2 \leq i, j \leq s$ such that $c'(v_i) = c'(v_j)$. This yields (d).

From (a)-(d) it follows that every edge v_iv_j with $r + 2 \leq i < j \leq s$ and $i \leq j - 2$ is such that

$$S(v_i) = \alpha\beta \text{ and } S(v_j) = \beta\gamma, \tag{2}$$

where $\{1, 2, 3\} = \{\alpha, \beta, \gamma\}$. Consequently, the path $v_r\text{-}\dots\text{-}v_{s-1}$ is a propagation path. By assumption, $|\{v_r, \dots, v_{s-1}\}| \leq \lambda$ and so $s - r \leq \lambda$.

A symmetric consideration holds for P^2. Let us now assume that $|V(P^1)| \geq |V(P^2)|$. It remains to show that r is bounded by some constant. To this end, recall that $\lambda \geq 20$.

Suppose that there is an edge v_iv_j with $3 \leq i \leq j \leq r$ such that $c'(v_i) = c'(v_j)$. We then put $x' = v_j$, $Q^1 = v_j\text{-}\dots\text{-}v_s$, and $Q^2 = v_j\text{-}\dots\text{-}v_i$. But this is a contradiction to the choice of x, P^1, and P^2, as $\max\{|V(Q^1)|, |V(Q^2)|\} < \max\{|V(P^1)|, |V(P^2)|\}$. In addition to the assertion we just proved, which corresponds to assertion (d) above, the assumptions (a)-(c) from above also hold here, where we replace r by 1 and s by r. Hence, using the same argumentation as above, we see that $r \leq \lambda + 1$. Summing up, we have $|V| \leq |V(P^1) \cup V(P^2)| \leq 2|V(P^1)| \leq 4\lambda + 2$, as desired.

Next suppose that $|V_1| = 1$. Now Claim 6 applies and we obtain the promised path, say $P = v_1\text{-}\dots\text{-}v_s$, with $|L(v_1)| = 1$. Let us say $L(v_1) = \{1\}$. Consider v_1 to be colored in color 1, and update along P, but only up to v_{s-1}. Due to the choice of P, the coloring so far is proper. Now when we update from v_{s-1} to v_s, two adjacent vertices recieve the same color. Let the partial coloring obtained so far be denoted c. Let X be the set of neighbors w of v_s on P with $c(w) = c(v_s)$, and let r be minimum such that $v_r \in X$. Moreover, let $c'(v_j)$ be the unique color in $L(v_j) \setminus \{c(v_j)\}$, for all $j = 2, \dots, s$. Just like in the case $|V_1| = 0$, we obtain the assertions (a)-(d) from above and this implies $s - r \leq \lambda$.

It remains to show that $r \leq \lambda + 1$. To see this, suppose that there is an edge $v_i v_j$ with $2 \leq i \leq j \leq r$ such that $c'(v_i) = c'(v_j)$. We then put $P^1 = v_j$-…-v_s and $P^2 = v_j$-…-v_i. Now, if we give color $c(v_j)$ to v_j and update along P^1 we obtain an improper coloring. Moreover, if we give color $c'(v_j)$ to v_j and update along P^2 we also obtain an improper coloring. This means that the pair $(G[V(P^1) \cup V(P^2)], L|_{V(P^1) \cup V(P^2)})$ is not colorable, in contradiction to the minimality of (G, L).

The assertion we just proved corresponds to assertion (d) above, and the assumptions (a)-(c) also hold here, where we replace r by 1 and s by r. Hence, we know $r \leq \lambda + 1$ and obtain $|V| = |V(P)| \leq 2\lambda + 1$.

Finally suppose that $|V_1| = 2$. Claim 7 applies and we obtain the promised path, say $P = v_1$-…-v_s, with $|L(v_1)| = |L(v_s)| = 1$. Let us say $L(v_1) = \{\alpha\}$ and $L(v_s) = \{\beta\}$. Consider v_1 to be colored in color α, and update along P, but only up to v_{s-1}. Due to the choice of P, the partial coloring so far is proper. Let the partial coloring obtained so far be denoted c, and put $c(v_s) = \beta$. We now have $c(v_{s-1}) = c(v_s)$, and this is the unique pair of adjacent vertices of G that receive the same color.

For each $j = 2, \dots, s-1$, we denote by $c'(v_j)$ the unique color in $L(v_j) \setminus \{c(v_j)\}$. We claim that s is bounded by a constant. Just like in the cases above the following assertions apply.

(a) $c(v_j) = c'(v_{j+1})$ for all $j = 1, \dots, s-2$.

(b) For every edge $v_i v_j$ with $1 \leq i, j \leq s-1$ it holds that $c(v_i) \neq c(v_j)$.

(c) For every edge $v_i v_j$ with $1 \leq i, j \leq s-1$ and $j - i \geq 2$ it holds that $c(v_i) \neq c'(v_j)$.

(d) For every edge $v_i v_j$ with $3 \leq i, j \leq s-1$ it holds that $c'(v_i) \neq c'(v_j)$.

Let $r' = 1$ and $s' = s - 1$. As above we see that $s' - r' \leq \lambda - 1$. Hence, $s \leq \lambda + 1$. From the fact that P is a Hamiltonian path in G we obtain the desired bound $|V| = |V(P)| \leq \lambda + 1$. This completes the proof. □

5 P_6-free minimal list obstructions

The aim of this section is to prove that there are only finitely many P_6-free minimal list-obstructions. In Section 5.1 we prove the following lemma which says that there are only finitely many P_6-free minimal list-obstructions with lists of size at most two.

Lemma 13. *Let (G, L) be a P_6-free minimal list-obstruction for which $|L(v)| \leq 2$ holds for all $v \in V(G)$. Then $|V(G)| \leq 100$.*

Our proof of this lemma is computer-aided. We use the computer to handle the enormous amount of case distinctions that appear in our proof. We do have a proof that works entirely by hand, but gives a significantly worse bound on the size of the obstructions.

In Section 5.2 we solve the general case, where each list may have up to three entries, making extensive use of Lemma 13. We prove the following lemma.

Lemma 14. *There are only finitely many P_6-free minimal list-obstructions.*

One of the main ideas in the proof of Lemma 14 is to guess the coloring on a small vertex subset S of the minimal list-obstruction at hand, (G, L) say. After successive updating, we arrive at a list-obstruction (G, L') where each list has size at most two, and so we may apply Lemma 13 to show that there is a minimal list-obstruction $(H, L'|_H)$ with a bounded number of vertices induced by (G, L'). We can prove that G is basically the union of these graphs H (one for each coloring of S), and so the number of vertices of G is bounded by a function of the number of guesses we took in the beginning. Since we precolor only a (carefully chosen) small part of the graph, we can derive that the number of vertices of G is bounded by a constant.

To find the right vertex set to guess colors for, we use a structure theorem for P_t-free graphs [4] that implies the existence of a well-structured connected dominating subgraph of a minimal list-obstruction.

5.1 Proof of Lemma 13

Let (G, L) be a P_6-free minimal list-obstruction such that every list contains at most two colors. Suppose that $P = v_1\text{-}\ldots\text{-}v_k$ is a propagation path in (G, L). We show that if G is P_6-free, then $k \leq 24$. According to Lemma 12, this proves that G has at most 100 vertices.

Our proof is computer-aided, but conceptually very simple. The program generates the paths v_1, $v_1 - v_2$, $v_1 - v_2 - v_3$, and so on, lists for each v_i, as in the definition of a propagation path, and edges among the vertices in the path. Whenever a P_6 or an edge violating condition (1) of the definition of a propagation path is found, the respective branch of the search tree is closed. Since the program does not find such a path on 25 vertices (cf. Table 2), our claim is proved.

Vertices	1	2	3	4	5	6	7	8	
Propagation paths	1	2	6	22	86	350	1 220	2 656	
Vertices	9	10	11	12	13	14	15	16	
Propagation paths	4 208	5 360	5 864	5 604	5 686	5 004	4 120	3 400	
Vertices	17	18	19	20	21	22	23	24	25
Propagation paths	2 454	1 688	1 064	516	202	72	18	2	0

Table 2: Counts of all P_6-free propagation paths with lists of size 2 meeting condition (1) generated by Algorithm 1.

The pseudocode of the algorithm is shown in Algorithm 1 and 2. Our implementation of this algorithm can be downloaded from [11]. Table 2 lists the number of configurations generated by our program.

Algorithm 1 Generate propagation paths and lists

$H \leftarrow (\{v_2\}, \emptyset)$
$c(v_1) \leftarrow 1$
$L(v_1) \leftarrow \{1\}$
Construct(H, c, L)
// We may assume $c(v_1) = 1$ and $L(v_1) = \{1\}$.

5.2 Proof of Lemma 14

We start with some claims to deal with vertices that have a special structure in their neighborhood.

Algorithm 2 Construct(Graph H, coloring c, list system L)

$j \leftarrow |V(H)|$
$V(H) \leftarrow V(H) \cup \{v_{j+1}\}$
$E(H) \leftarrow E(H) \cup \{v_j v_{j+1}\}$
    // This extends the path by the next vertex v_{j+1}.
for all $\alpha \in \{1,2,3\} \setminus \{c(v_j)\}$ and all $I \subseteq \{1, 2, \ldots, j-1\}$ **do**
    $H' \leftarrow H$
    $E(H') \leftarrow E(H') \cup \{v_i v_{j+1} : i \in I\}$
        // This adds edges from v_{j+1} to earlier vertices in all possible ways.
    $c(v_{j+1}) \leftarrow \alpha$
    $L(v_{j+1}) \leftarrow \{\alpha, c(v_j)\}$
    if (H', c, L) is P_6-free and satisfies condition (1) **then**
        Construct(H', c, L)
            // If the propagation path is not pruned, we extend it further.
    end if
end for

Claim 8. *Let G be a P_6-free graph and let $v \in V(G)$. Suppose that $G[N(v)]$ is a connected bipartite graph with bipartition classes (A, B). Let G' be obtained from $G \setminus (A \cup B)$ by adding two new vertices a, b with $N_{G'}(a) = \{b\} \cup \bigcup_{u \in A}(N_G(u) \cap V(G'))$ and $N_{G'}(b) = \{a\} \cup \bigcup_{u \in B}(N_G(u) \cap V(G'))$. Then G' is P_6-free.*

Proof. Suppose Q is a P_6 in G. Then $V(Q) \cap \{a, b\} \neq \emptyset$. Observe that if both a and b are in $V(Q)$, then $v \notin V(Q)$. If only one vertex of Q, say q, has a neighbor in $\{a, b\}$, say a, then we get a P_6 in G by replacing a with a neighbor of q in A, and, if $b \in V(Q)$, replacing b with v. Thus we may assume that two vertices q, q' of Q have a neighbor in $\{a, b\}$. If q and q' have a common neighbor $u \in A \cup B$, then $G[(V(Q) \setminus \{a, b\} \cup \{u\})]$ is a P_6 in G, a contradiction. So no such u exists, and in particular $v \notin V(Q)$. Let Q' be an induced path from q to q' with $V(Q') \setminus \{q, q'\} \subseteq A \cup B \cup \{v\}$, meeting only one of the sets A, B if possible. Then $G[(V(Q) \setminus \{a, b\}) \cup V(Q')]$ is a P_6 in G, a contradiction. This proves Claim 8. □

In the remainder of this section, let (G, L) be a P_6-free minimal list-obstruction.

Claim 9. *Let H be an induced subraph of G such that H is connected and bipartite. Let (A, B) be the bipartition of H, and let $u \in V(G)$ be complete to $A \cup B$. Let $K = V(G) \setminus (A \cup B)$ and $k = |K|$. Then $|A| \leq 7 \cdot 2^{7 \cdot 2^k + k}$.*

Proof. We partition A according to the adjacency in K and the lists of the vertices. More precisely, let $A = A_1 \cup A_2 \cup \ldots \cup A_{7 \cdot 2^k}$ such that for any i and for any $x, y \in A_i$, $N(x) \cap K = N(y) \cap K$ and $L(x) = L(y)$. Analogously, let $B = B_1 \cup B_2 \cup \ldots \cup B_{7 \cdot 2^k}$.

Next, for $i = 1, \ldots, 7 \cdot 2^k$, we partition $A_i = A_i^1 \cup \ldots \cup A_i^{7 \cdot 2^k}$ according to the neighbors in B. More precisely, for any t and any $x, y \in A_i^t$, $N(x) \cap B_j \neq \emptyset$ if and only if $N(y) \cap B_j \neq \emptyset$, $j = 1, \ldots, 7 \cdot 2^k$. We partition the sets in B analogously. It is sufficient to prove the following.

$$\textit{There do not exist } x, y \in A_i^j \textit{ such that } x \neq y. \tag{3}$$

Suppose that there exist $x, y \in A_i^j$ with $x \neq y$. Let $N = N(x) \cap B$. Let C be the component of $H \setminus x$ with $y \in C$. Let N_1 be the set of vertices of N whose unique neighbor in H is x, and let $N_2 = N \setminus N_1$.

Suppose first that there is a vertex $s \in N_2 \setminus C$. Since y is not dominated by x, there exists $t \in (B \cap N(y)) \setminus N(x)$. Since $s \notin C$, it follows that s and t have no common neighbor in H, and so, since G is P_6-free, there is a 5-vertex path Q in H with ends s and t; let the vertices of Q be q_1-…-q_5, where $s = q_1$ and $t = q_5$. Since $s \notin C$, it follows that $q_2 = x$. Since y-t-q_4-q_3-x-s is not a P_6 in G, it follows that y is adjacent to q_3, and so we may assume that $q_4 = y$. Let $p \in A \setminus \{x\}$ be a neighbor of s. Since p-s-x-q_3-y-t is not a P_6, it follows that p has a neighbor in $\{t, q_3\}$, contrary to the fact that $s \notin C$. This proves that $N_2 \subseteq C$.

Observe that $C \cap N_1 = \emptyset$. By the minimality of (G, L), there is a coloring c of $(G \setminus (N_1 \cup \{x\}), L|_{G \setminus (N_1 \cup \{x\})})$. Since u is complete to $V(C)$, it follows that $C \cap A$ and $C \cap B$ are both monochromatic. We now describe a coloring of G. Color x with $c(y)$. Let $n_1 \in N_1$ be arbitrary. Since $x, y \in A_i^j$, there exists n_1' in B such that n_1' is adjacent to y, $L(n_1) = L(n_1')$, and n_1, n_1' have the same neighbors in K. Now color n_1 with $c(n_1')$. Repeating this for every vertex of N_1 produces a coloring of (G, L) a contradiction. This proves (3), and Claim 9 follows. □

Claim 10. *Let $v \in V(G)$ with $N(v) = A \cup B$ where $A = \{a_1, \ldots, a_s\}$ and $B = \{b_1, \ldots, b_s\}$. Assume that $N(a_i) \cap N(v) = B \setminus \{b_i\}$ and $N(b_i) \cap N(v) = A \setminus \{a_i\}$ for all $i \in \{1, \ldots, s\}$. Let $K \subseteq V(G) \setminus (N(v) \cup \{v\})$ be the set of vertices mixed on $A \cup B$, and let $k = |K|$. Then $s \leq 7 \cdot 2^k$.*

Proof. We may assume that $s \geq 3$. We partition A according to the adjacency in K. More precisely, let $A = A_1 \cup A_2 \cup \ldots \cup A_{2^k}$ such that for any i and for all $x, y \in A_i$, $N(x) \cap K = N(y) \cap K$. From the minimality of (G, L) it is sufficient to proof that there do not exist two vertices $x, y \in A^j$ with $L(x) = L(y)$. Suppose for a contradiction that such vertices x, y exist. By the minimality of G, there is an L-coloring c of $G \setminus x$. Note that both $A \setminus x$ and B are monochromatic in c. Since $N(x) \subseteq N(y) \cup B$ and $c(y)$ is different from the color of B, c can be changed to a L-coloring of G by setting $c(x) = c(y)$, a contradiction. This completes the proof. □

Claim 11. *Let $x \in V(G)$ such that $L(x) = \{1, 2, 3\}$ and let $U, W \subseteq V(G)$ be disjoint non-empty sets such that $N(x) = U \cup W$ and U is complete to W. Then (possibly exchanging the roles of U and W) there is a path P with $|V(P)| = 4$, such that the ends of P are in U, no internal vertex of P is in U, and $V(P) \cap W = \emptyset$.*

Proof. Since we may assume that $G \neq K_4$, it follows that U and W are both stable sets. By the minimality of G, there is an L-coloring of $G \setminus x$, say c. Since G does not have an L-coloring, we may assume that there exist $u_1, u_2 \in U$ such that $c(u_i) = i$. Then $c(w) = 3$ for every $w \in W$, and $c(u) \in \{1, 2\}$ for every $u \in U$. For $i = 1, 2$ let $U_i = \{u \in U : c(u) = i\}$. Let $V_{12} = \{v \in V(G) : c(v) \in \{1, 2\}\}$, and let $G_{12} = G[V_{12}]$. If no component of G_{12} meets both U_1 and U_2, then exchanging colors 1 and 2 on the components that meet U_1 produces a coloring of $G \setminus x$ where every vertex of U is colored 2 and every vertex of W is colored 3; this coloring can then be extended to G, which is a contradiction. This proves that some component C of G_{12} meets both U_1 and U_2.

Let P be a shortest path from U_1 to U_2 in G_{12}. Then $|V(P)| = 4$ since G is P_6-free. Moreover, $V(P) \cap W = \emptyset$, and no interior vertex of P is in U. □

We now make use of the following result.

Theorem 15 (Camby and Schaudt [4])**.** *For all $t \geq 3$, any connected P_t-free graph H contains a connected dominating set whose induced subgraph is either P_{t-2}-free, or isomorphic to P_{t-2}.*

Applying Theorem 15 to a connected P_6-free graph, one obtains a connected dominating set whose induced subgraph is either P_4 or P_4-free. In the first case, we are done by applying Lemma 9 and 13. Hence for the remainder of this section, we assume that G contains a dominating connected

P_4-free induced subgraph H. Recall that every P_4-free graph is a *cograph* [6]. Thus, $V(H) = A \cup B$ where A is complete to B. We may assume that both A and B are non-empty, since otherwise G has a dominating vertex, namely the vertex contained in H. In this case, however, we are done by Lemma 9. We may assume, moreover, that every vertex of G complete to B is in A, and every vertex complete to A is in B. Let $K = V(G) \setminus (A \cup B)$.

Our strategy from now on is as follows. At every step, we find a small subgraph R of G, and consider all possible precolorings of R. For each precoloring, (G, L) contains a minimal list-obstruction (G', L') and by Lemma 7 it is enough to consider each of these minimal list-obstructions separately. We update three times with respect to $V(R)$, and show that in all cases $|L'(v)| \leq 2$ for every $v \in V(G')$, and so by Lemma 13 $V(G') \setminus R$ has bounded size. It follows from Lemma 7 that $V(G) \setminus R$ has bounded size. Now we use the minimality of G and the internal structure of R to show that R also has a bounded number of vertices, and so $|V(G)|$ is bounded.

Claim 12. *If G is C_5-free, $|V(G)| \leq 2^{10^7}$.*

Proof. It is clear that in any possible coloring of H, either A or B is monochromatic. We first prove the following

$$\text{If both } A \text{ and } B \text{ are stable, then } |A| \leq 7 \cdot 2^{|K|}. \tag{4}$$

To see this, we write $|K| = k$. Partition A by the adjacency in K; more precisely let $A = A_1 \cup A_2 \cup \ldots \cup A_{2^k}$ such that for any i and for any $x, y \in A_i$, $N(x) \cap K = N(y) \cap K$. Similarly partition $B = B_1 \cup B_2 \cup \ldots \cup B_{2^k}$. We claim that there does not exist $x, y \in A_i$, with $L(x) = L(y)$. To see this, suppose that such x, y exist. By the minimality of G, $G \setminus x$ is L-colorable. As $N(x) = N(y)$, G is L-colorable by giving x the same color as y, a contradiction. This proves (4).

Next we define an induced subgraph R of G, and a refinement $\mathcal{L}$ of L with respect to R. Suppose first that there is an edge uv with $u, v \in A$ or $u, v \in B$. Let $R = G[\{u, v\}]$. For every $L|_R$-coloring c of R, let $L_c(u) = c(u)$, $L_c(v) = c(v)$ and $L_c(w) = L(w)$ for every $w \in V(G) \setminus \{u, v\}$. Let $\mathcal{L}$ be the set of all such lists L_c. Now assume that each of A, B is stable. Let $R = H$ and let $\mathcal{L}$ consist of the following two types of list systems.

For each $i \in \bigcap_{a \in A} L(a)$ we add the list system L' to $\mathcal{L}$, where $L'(a) = \{i\}$ for all $a \in A$ and $L'(v) = L(v)$ for all $v \in V(G) \setminus A$. Moreover, for each $j \in \bigcap_{b \in B} L(b)$ we add the list system L' to $\mathcal{L}$, where $L'(b) = \{j\}$ for all $b \in B$ and $L'(v) = L(v)$ for all $v \in V(G) \setminus B$.

It is clear that the system $\mathcal{L}$ defined in such a way is a refinement of L and satisfies the hypotheses of Lemma 7. By (4), Lemma 7 and symmetry, we may assume that $L(b) = \{1\}$ for every $b \in B$. Let X be the set of vertices that have lists of size 3 after updating three times with respect to B. In particular, $A \cap X = \emptyset$. In particular, $A \cap X = \emptyset$ and X is anticomplete to B.

$$\text{Let } x \in X \text{ and let } U, W \subseteq V(G) \text{ be disjoint non-empty sets such that } N(x) = U \cup W. \text{ Then } U \text{ is not complete to } W. \tag{5}$$

If U is complete to W, there exists a path P as in Claim 11. But then $G[V(P) \cup \{v\}]$ is a C_5, a contradiction. This proves (5).

Let $\{Y_1, Y_2, \ldots, Y_n\}$ be the set of components of A that are not anticomplete to X. Let Y be a minimum size subset of $\{Y_1, Y_2, \ldots, Y_n\}$ with the property that the set $\bigcup_{Y_i \in Y} Y_i$ contains a neighbor of every vertex of X. For simplicity, we may assume $Y = \{Y_1, Y_2, \ldots, Y_m\}$. As all vertices in Y_i have list $\{2, 3\}$ after updating, each component has two possible colorings. If $m \leq 2$, considering all possible colorings of $G[(Y_1 \cup Y_2)]$ and updating with respect to $Y_1 \cup Y_2$ for each coloring, then

Claim 12 follows from (4), Lemma 13, Lemma 7, and Lemma 8. So we may assume that $m \geq 3$. Now fix a precoloring c of $G|(Y_1 \cup Y_2 \cup Y_3)$ and update three times with respect to $Y_1 \cup Y_2 \cup Y_3$. Let X' be the set of vertices that still have list of size 3 after this updating, and let $x \in X'$. Since x is not dominated by a vertex in B, $N(x) \setminus A \neq \emptyset$. By (5), $N(x) \cap A$ is not complete to $N(x) \setminus A$. Let $v \in N(x) \cap A$ and $u \in N(x) \setminus A$ such that uv is not an edge, and let V be the component of A that contains v. Then $V \neq Y_1, Y_2, Y_3$. We can choose $b \in B$ such that bu is not an edge since every vertex complete to B is in A.

There exist distinct $i, j \in \{1,2,3\}$ and vertices $y_i \in Y_i$, $y_j \in Y_j$, $x_i, x_j \in X$, such that $x_i y_i$ is an edge and vx_i is a non-edge. (6)

Suppose this is false. For $\{i,j,k\} = \{1,2,3\}$ let X_i be the set of vertices in X that only have neighbors in Y_i, and let X_{ij} be the set of vertices in X that have neighbors in Y_i and Y_j, but not in Y_k. By the choice of Y, we know that X_i is nonempty for every $i \in \{1,2,3\}$. We may assume that v is complete to X_1, X_2, for otherwise we can choose $x_i \in X_i$ and $y_i \in Y_i$ with $i = 1,2$ as required. If v is complete to X_{12}, then $(Y \setminus \{Y_1, Y_2\}) \cup \{V\}$ contradicts the choice of Y, so we can choose $x_1 \in X_{12}$ non-adjacent to v. Similarly, v has a non-neighbor x_3 in $X_3 \cup X_{23}$. Now, for $i \in \{1,3\}$ let $y_i \in Y_i$ be a neighbor of x_i. This proves (6).

X' is empty. (7)

Suppose this is false and choose $x \in X'$ and $v, u, b, y_i, y_j, x_i, x_j$ as described above. We may assume that $i = 1$ and $j = 2$. Then $\{x, x_1, x_2\}$ is anticomplete to B. Moreover, the set $\{v, y_i, y_j\}$ is a stable set as those vertices belong to different components of A. Since G does not contain an induced C_5, the set $\{x, x_i, x_j\}$ is also a stable set. Now since, for $i \in \{1,2\}$, u-x-v-b-y_i-x_i is not an induced P_6, it follows that u is adjacent to y_i or x_i. If uy_i is an edge, then b-y_i-u-x-v-b is an induced C_5, a contradiction. Therefore, u is anticomplete to $\{y_1, y_2\}$ and u is complete to $\{x_1, x_2\}$. But now x-u-x_1-y_1-b-y_2 is an induced P_6, a contradiction. This proves (7).

It follows from (7) that all vertices now have lists of size at most 2. Thus, Lemma 13, Lemma 7 and Lemma 8 together imply that $|K| \leq 6 \cdot 36 \cdot 8 \cdot 36 \cdot 100$. Therefore, (4) implies $|V(G)| \leq 2^{10^7}$, as desired. □

In the remainder of this proof we deal with minimal list-obstructions containing a C_5, by taking advantage of the structure that it imposes. We assume, throughout the remainder of this section, that there is a C_5 in G, say $C = c_1$-c_2-c_3-c_4-c_5-c_1. Let $X(C)$ be the set of vertices of $V(G) \setminus V(C)$ that have a neighbor in C, let $Y(C)$ be the set of vertices of $V(G) \setminus (V(C) \cup X(C))$ that have a neighbor in X, and let $Z(C) = V(G) \setminus (V(C) \cup X(C) \cup Y(C))$.

Claim 13. *Assume that $|V(G)| \geq 7$. Then the following assertions hold.*

1. *For every $x \in X(C)$ there exists indices $i, j \in \{1, \ldots, 5\}$ such that x-c_i-c_j is an induced path.*
2. *No vertex of $Y(C)$ is mixed on an edge of $G[Z(C)]$.*
3. *If $v \in X(C)$ is mixed on an edge of $G[Y(C) \cup Z(C)]$, then the set of neighbors of v in C is not contained in a 3-vertex path of C.*
4. *If $v \in X(C)$ has a neighbor in $Y(C)$, then the set of neighbors of x in C is not contained in a 2-vertex path of C.*

Proof. Since $|V(G)| \geq 7$, no vertex is adjacent to all vertices of C, as that would lead to a list-obstruction on 6 vertices. Thus, the first assertion follows from the fact that G is connected.

Next we prove the second assertion. Suppose that $u \in Y(C)$ is mixed on the edge st with $s,t \in Z(C)$, namely u is adjacent to s and not to t. Let $b \in N(u) \cap X$ and $i,j \in \{1,\dots,5\}$ be such that b-c_i-c_j is an induced path. Then t-s-u-b-c_i-c_j is a P_6, a contradiction.

To see the third assertion, suppose that $x \in X$ is adjacent to $t \in Y$ and non-adjacent to $s \in Y \cup Z$, where t is adjacent to s, and suppose that $N(x) \cap V(C) \subseteq \{c_1, c_2, c_3\}$. We may assume that x is adjacent to c_3. Then c_5-c_4-c_3-x-t-s is a P_6 in G, a contradiction.

To prove the fourth statement, we may assume that $x \in X$ is adjacent to c_1 and to $y \in Y$, and non-adjacent to c_2, c_3 and c_4. Now y-x-c_1-c_2-c_3-c_4 is a P_6 in G, a contradiction. □

Claim 14. *Let (G,L) be obtained from G by precoloring the vertices of C and updating the lists three times, then at least one of the following holds:*

1. *For every $v \in Z(C)$, $|L(v)| \leq 2$, or*
2. *there exists a minimal obstruction, $(G', L|_{G'})$ induced by (G,L) such that $|V(G')| \leq 8$.*

Proof. We write $X = X(C)$, $Y = Y(C)$ and $Z = Z(C)$. Suppose for a contradiction that there is a vertex $v \in Z$ with $|L(v)| = 3$ and that every minimal obstruction induced by (G,L) contains at least 9 vertices. Let D be the component of Z containing v. By Claim 13.2, no vertex of $V(G) \setminus Z$ is mixed on D, and it follows from the minimality of G that $G[D]$ is bipartite.

If $z \in Z$ and $u,t \in N(z) \cap Y$ are non-adjacent, then no vertex of X is mixed on $\{u,t\}$. (8)

Suppose that $w \in X$ is adjacent to u and non-adjacent to v. By 13.1 there exists i,j such that w-c_i-c_j is an induced path. Then t-z-u-w-c_i-c_{i+1} is a P_6, a contradiction. This proves (8).

$N(v) = U \cup W$ where (9)

- *both U and W are non-emtpy,*
- *U is complete to W, and*
- *$U \subseteq Y$, and either $W \subseteq Y$ or $W \subseteq D$.*

First we assume that $D = \{v\}$, and thus $N(v) \subseteq Y$.

If $N(v)$ is a connected set, we proceed as follows. Pick $u \in N(v)$ and $w \in N(u) \cap X$. Let $U = N(v) \cap N(w)$ and $W = N(v) \setminus N(w)$. Since v is not dominated by w, U and W are both non-empty. By (8), U is complete to W and we are done.

So, we may assume that $N(v)$ is disconnected, and therefore anticonnected. Let $w \in X$ have a neighbor in $N(v)$. Since w does not dominate v, it follows that w is mixed on $N(v)$, and since $N(v)$ is anticonnected there exist $t,u \in N(v)$ such that w is adjacent to u and not to t, and u is non-adjacent to t, contrary to (8).

Thus we may assume that $|D| \geq 2$. Recall that $G[D]$ is bipartite and let D_1, D_2 be the bipartition. Let U be the set of vertices of Y with a neighbor in D. By Claim 13.2, U is complete to D. We may assume that $v \in D_1$; set $W = D_2$. This proves (9).

Let U, W be as in (9) and let P be a path as in Claim 11. We first argue that we may assume that the ends of P are in U. If $|D| = 1$, this is true by symmetry. If $|D| \geq 2$, note that W is anticomplete to $V(G) \setminus (D \cup U)$. Since P contains only four vertices, P must be entirely contained in $G[D]$, and yet P is a 4 vertex path joining two vertices on the same side of the bipartition, a contradiction. This proves that the ends of P are in U.

Let $P = p_1 - p_2 - p_3 - p_4$. By (8), $p_2, p_3 \notin X$, and so $p_2, p_3 \in Y \cup Z$. By Claim 13.2 we may assume that $p_2 \in Y$. Let $x_2 \in X$ be adjacent to p_2, and let i be such that $x_2 - c_i - c_{i+1}$ is a path (such i exists by Claim 13.1). If x_2 is non-adjacent to p_1, then $v - p_1 - p_2 - x_2 - c_i - c_{i+1}$ is a P_6, a contradiction. So x_2 is adjacent to p_1, and therefore, by (8), x_2 is adjacent to p_4. Since x_2 is mixed on an edge of $Y \cup Z$, Claim 13.3 implies that x_2 has two consecutive neighbors in C. So $|L(x_2)| = 1$ since we have updated. We may assume that $L(x_2) = \{1\}$. Let w be a vertex in W. If $1 \notin L(p_3)$, then $G[\{p_1, p_2, p_3, p_4, x_2, v, w\}]$ is not L-colorable, contrary to that every minimal obstruction induced by (G, L) contains at least 9 vertices. This proves that $1 \in L(p_3)$, and, since we have updated three times with respect to $V(C)$, x_2 is non-adjacent to p_3. Since x_2 does not dominate p_3, it follows that p_3 has a neighbor x_3 non-adjacent to x_2. Suppose first that $x_3 \notin X$. Then x_3 is anticomplete to $\{c_i, c_{i+1}\}$. Since $x_3 - p_3 - p_2 - x_2 - c_i - c_{i+1}$ and $x_3 - p_3 - p_4 - x_2 - c_i - c_{i+1}$ are not copies P_6 in G, it follows that x_3 is complete to $\{p_2, p_4\}$. But now $G[\{p_1, p_2, p_3, p_4, x_2, x_3, v, w\}]$ is not L-colorable, contrary to that every minimal obstruction induced by (G, L) contains at least 9 vertices. This proves that $x_3 \in X$ (and therefore $p_3 \in Y$). Now there is symmetry between (p_2, x_2) and (p_3, x_3). It follows that x_3 is complete to $\{p_1, p_3, p_4\}$ and non-adjacent to p_2, and that $|L(x_3)| = 1$. Since $1 \in L(p_3)$, it follows from the fact that we have updated three times with respect to $V(C)$ that $L(x_3) \neq L(x_2)$. This implies that $|L(p_1)| = 1$, and so $|L(v)| \leq 2$ (again since we have updated three times), a contradiction. This completes the proof. □

Claim 15. *Let $A = \{a \in X(C) : N(a) \cap V(C) = \{c_2, c_5\}\}$ and $W = \{y \in Y : N(y) \cap X \subseteq A\}$. Assume that we properly precolor the vertices of C so that c_2 and c_5 receive the same color, say j. We update with respect to C three times and delete all non-critical vertices in W. Let D be a component of W such that there exists a vertex with list of size 3 in D, and let N be the set of vertices of A with a neighbor in D. Then D is complete to N, and either*

- *D is anticomplete to $V(G) \setminus (D \cup N)$, or*
- *there exists vertices $d \in D$ and $v \in N(d)$ such that precoloring d, v and updating with respect to the set $\{d, v\}$ three times reduces the list size of all vertices in W to at most two.*

Proof. By Claim 13.3 no vertex of $X(C)$ is mixed on D, and so N is complete to D. Also by Claim 13.3 W is anticomplete to $Z(C)$.

First suppose that some $d \in D$ has a neighbor $v \notin N \cup D$ such that there exists $x \in N(v) \setminus A$ with $N(x) \cap \{c_2, c_5\} \neq \emptyset$. Since $D \subseteq W$, it follows that $v \in Y(C)$. By Claim 13.3, $N(v) \cap A = N$ and $N(x) \cap C$ are not contained in a 3-vertex path of C. Thus $|L(x)| = 1$ and $j \notin L(x)$. We precolor $\{v, d\}$ and update with respect to the set $\{v, d\}$ three times.Since $\{v, d\}$ is complete to N and we may assume that not both v, d are precolored j, it follows that $|L(n)| = 1$ for every $n \in N$. Consequently $|L(u)| \leq 2$ for every $u \in W$ such that u has a neighbor $n \in N$ where n is complete to $\{v, d\}$. Suppose there is $t \in W$ with $|L(t)| = 3$. Then t is anticomplete to $\{v, d\}$. Since $t \in W$, there exists $s \in A$ adjacent to t, and so s is not complete to $\{v, d\}$. Since $s \in A$, it follows from Claim 15.3 that s is not mixed on the edge vd, and so s is anticomplete to $\{v, d\}$. Since $|L(t)| = 3$, it follows that $L(s) = \{1, 2, 3\} \setminus \{j\}$, and so s is non-adjacent to x (since we have updated three times with respect to $V(C)$). Assume by symmetry that c_2 is adjacent to x, then $t - s - c_2 - x - d - v$ is a P_6, a contradiction.

Therefore we may assume that we cannot find such $\{d, v\}$. Suppose there still exists a vertex $e \in D$ that has a neighbor $w \notin (N \cup D)$. Then again by Claim 13.3, $N(w) \cap A = N$ and, for all $y \in N(w) \cap X(C) \setminus A$, the set $N(y) \cap C$ is not contained in a 3-vertex path of C. By our assumption

$N(y) = \{c_1, c_3, c_4\}$, and hence $L(y) = \{j\}$ and thus $L(w) \subseteq \{1,2,3\} \setminus \{j\}$. If $D = \{e\}$, then $|L(e)| = 3$; but $L(u) \subseteq \{1,2,3\} \setminus \{j\}$ for all $u \in N(e)$, which contradicts the fact that e is critical. Therefore we may assume there exists $f \in N(e) \cap D$. Since G is not a K_4, f is not adjacent to w. But now c_5-c_1-y-w-e-f is a P_6, a contradiction. □

Claim 16. *Assume that there is a vertex $c_1' \in V(G)$ adjacent to c_1, c_2, c_5 and non-adjacent to c_3, c_4. Then $|V(G)| \leq 2^{2^{104}}$.*

Proof. We may assume that $|V(G)| \geq 8$. Precolor the vertices of C and update with respect to C three times. By symmetry, we may assume that $L(c_1) = \{1\}$, $L(c_1') = L(c_3) = \{2\}$, $L(c_2) = L(c_5) = \{3\}$ and $L(c_4) = \{1\}$. Let $C' = c_1'$-c_2-c_3-c_4-c_5-c_1'. We write $X = X(C)$, $X' = X(C')$, and define the sets Y, Y', Z, and Z' in a similar manner.

Let A be the set of all vertices $a \in X \cup X'$ for which $N(a) \cap \{c_1, c_1', c_2, c_3, c_4, c_5\} = \{c_2, c_5\}$. Let W be the set of vertices $y \in Y \cap Y'$ such that $N(y) \cap (X \cup X') \subseteq A$. Since we have updated $|L(x)| \leq 2$ for every $x \in X \cup X'$, and, by Claim 14, $|L(z)| \leq 2$ for every $z \in Z \cup Z'$. Thus if $|L(v)| = 3$ then $v \in Y \cap Y'$, and easy case analysis shows that $v \in W$. We may assume that all vertices in W are critical. By Lemma 13 we may assume that $W \neq \emptyset$.

> *Let D be a component of W, and let N be the set of vertices in A with a neighbor in D. Then D is complete to N and we may assume D is anticomplete to $V(G) \setminus (D \cup N)$.* (10)

(10) follows from Claim 15 readily.

Since $|V(G)| > 6$, it follows that every component D of W is bipartite. Let $D_1, \ldots, D_k$ be the components of W that contain a vertex v with $|L(v)| = 3$.

Suppose that $|D_i| = \{d\}$ for some i. Then, letting c be a coloring of $G \setminus d$, we observe that no vertex of $N(d)$ is colored 3, and so we can get a coloring of G by setting $c(d) = 3$, a contradiction.

So for every i, $|D_i| \geq 2$ and $G[D_i]$ is bipartite. Let (A_i, B_i) be the bipartition of $G[D_i]$, where $|\bigcap_{a \in A_i} L(a)| = 3$ if possible. We may assume that $|\bigcap_{a \in A_i} L(a)| = 3$ for $A_1, \ldots, A_t$ and $|\bigcap_{a \in A_i} L(a)| < 3$ for $A_{t+1}, \ldots, A_k$. Then, by the choice of A_i, $|\bigcap_{b \in B_i} L(b)| < 3$ for $B_{t+1}, \ldots, B_k$. Assume further that $3 \in \bigcap_{b \in B_i} L(b)$ for $B_1, \ldots, B_s$ and $3 \notin \bigcap_{b \in B_i} L(b)$ for $B_{s+1}, \ldots, B_t$. For every $i \in \{1, \ldots, s\}$ choose $a_i \in A_i$, for every $i \in \{s+1, \ldots, t\}$ choose $b_i \in B_i$, and for every $i \in \{t+1, \ldots, k\}$ choose a pair of adjacent vertices $a_i, b_i \in D_i$. Let

$$G' = G \setminus (\bigcup_{i=1}^{s} (D_i \setminus a_i) \cup (\bigcup_{i=s+1}^{t} (D_i \setminus b_i) \cup \bigcup_{i=t+1}^{k} (D_i \setminus \{a_i, b_i\})).$$

For $1 \leq 1 \leq s$ let

$$L'(a_i) = \{1, 2\},$$

for $s+1 \leq i \leq t$ let

$$L'(b_i) = \bigcap_{b \in B_i} L(b),$$

and for $t+1 \leq i \leq k$ let

$$L'(a_i) = \bigcap_{a \in A_i} L(a)$$

$$L'(b_i) = \bigcap_{b \in B_i} L(b).$$

Let $L'(v) = L(v)$ for every other $v \in V(G')$. (10) implies that every L'-coloring of G' can be converted to an L-coloring of G (as follows: for every $i \in \{1,\ldots,s\}$ assign every vertex of B_i the color 3, for $i \in \{s+1,\ldots,k\}$ assign every vertex of B_i the color of b_i; for $i \in \{s+1,\ldots,t\}$ assign every vertex of A_i the color 3, and for $i \in \{1,\ldots,s,t+1,\ldots,k\}$ assign every vertex of A_i the color of a_i), and therefore G' is not L'-colorable. Moreover, $|L'(v)| \leq 2$ for every $v \in V(G')$. Now Lemma 13 implies that G' contains an induced subgraph G'' with at most 100 vertices such that $(G'', L'|_{V(G'')})$ is not colorable. We may assume that for each $i \leq k$, $a_i \in G''$ or $b_i \in G''$, since otherwise we can delete D_i from G.

$$F = G[(V(G) \cap V(G'')) \cup (D_1 \cup \ldots D_k)] \textit{ is not } L|_F\textit{-colorable.} \tag{11}$$

Suppose that F is $L|_F$-colorable and let c be an $L|_F$-coloring of F. Observe that each of the sets $A_1,\ldots,A_k,B_1,\ldots,B_k$ is monochromatic. For every $s+1 \leq i \leq k$ the unique color that appears on B_i belongs to $L'(b_i)$, and for every $t+1 \leq i \leq k$ the unique color that appears on A_i belongs to $L'(a_i)$. Moreover for every $i \leq s$, either the unique color that appears on A_i belongs to $L'(a_i)$. This implies that c can be converted to an L'-coloring G'', a contradiction. This proves (11).

By (11), $|F \setminus (D_1 \cup \ldots D_k)| + k \leq 100$. Now applying Claim 9 k times we deduce that $|F| \leq 100 + 100 \cdot 14 \cdot 2^{7\cdot 2^{100}+100}$. Hence there exists a minimal list-obstruction induced by (G, L) with at most $100 + 100 \cdot 14 \cdot 2^{7\cdot 2^{100}+100}$ vertices. By Lemma 7 and Lemma 8, $V(G) \leq 3^6 \cdot 36 \cdot (100 + 100 \cdot 14 \cdot 2^{7\cdot 2^{100}+100}) \leq 10^6 \cdot 2^{7\cdot 2^{100}+100} \leq 2^{2^{104}}$. This completes the proof. □

We can now prove the following claim, which is the last step of our argument.

Claim 17. *If G contains a C_5, then $|V(G)| \leq 2^{2^{104}}$.*

Proof. We may assume that $|V(G)| \geq 9$. Applying Lemma 7, we may assume $|L(c_i)| = 1$ for all $i \in \{1,\ldots,5\}$ and, without loss of generality, $L(c_1) = 1$, $L(c_2) = L(c_4) = 2$, $L(c_3) = L(c_5) = 3$. By Lemma 8 we may assume that we have updated the list with respect to C three times and consider (G, L) to be a minimal list-obstruction. Write $X = X(C), Y = Y(C)$ and $Z = Z(C)$. Moreover, let $A' = \{v \in X : N(v) \cap C = \{c_2, c_4\}\}$ and $B' = \{v \in X : N(v) \cap C = \{c_3, c_5\}\}$.

It is immediate that $|L(v)| = 1$ for every vertex $v \in X \setminus (A' \cup B')$ that has a neighbor in Y. Let Y' be the set of vertices that have lists of size 3. Claim 14 implies that $Y' \subseteq Y$. Moreover, all vertices in Y' are critical since (G, L) is a minimal obstruction.

Let A, B be the subsets of A', B' respectively consisting of all vertices with a neighbor in Y'. Then the list of every vertex in A is $\{1, 3\}$ and the list of every vertex in B is $\{1, 2\}$. If one of A, B is not a stable set, Claim 16 completes the proof. So, we may assume that each of A, B is a stable set.

Let H be the graph obtained from $G[A \cup B]$ by making each of A, B a clique. Let $C_1,\ldots,C_t$ be the anticomponents of H such that both $A_i = C_i \cap A$ and $B_i = C_i \cap B$ are nonempty. Let $A'' = A \setminus \bigcup_{i=1}^t C_i$ and let $B'' = B_i \setminus \bigcup_{i=1}^t C_i$.

Let $v \in Y'$. Then $N(v) \cap A$ is complete to $B' \setminus N(v)$, and $N(v) \cap B$ is complete to $A' \setminus N(v)$. In particular, A is complete to $B' \setminus B$, B is complete to $A' \setminus A$, and v is not mixed on C_i for any i. (12)

Suppose this is false. By symmetry, we may assume there exists $k \in A_i$ non-adjacent to $w \in B' \setminus N(v)$ such that v is adjacent to w but not to k. Then v-w-c_5-c_1-c_2-k is a P_6, a contradiction. This proves (12).

$$\textit{Let } v \in Y'. \textit{ Then } N(v) \subseteq A \cup B. \tag{13}$$

We know that $N(v) \cap X \subseteq A \cup B$ as $|L(w)| = 1$ for all $w \in X \setminus (A' \cup B')$ and we have updated. Suppose that $N(v) \setminus X \neq \emptyset$. It follows from Claim 13.3 that $N(v) \cap X$ is complete to $N(v) \setminus X$.

Suppose first that v has both a neighbor in A' and a neighbor in B'. Let $y \in N(v) \setminus X$. We precolor v and y (in addition to C) and update three times. We claim that, after updating, $|L(x)| = 1$ for every $x \in X$. Recall that even before we precolored v and y we had that $|L(x)| = 1$ for every $x \in X \setminus (A \cup B)$. Since v and y are colored with different colors, and $\{v, y\}$ is complete to $N(v) \cap X$, it follows that $|L(x)| = 1$ for every $x \in N(v) \cap X$, and we may assume that $L(x) = \{1\}$ for every $x \in N(v) \cap X$. By (12), $N(v) \cap A$ is complete to $B \setminus N(v)$, and $N(v) \cap B$ is complete to $A \setminus N(v)$. Since we have updated, $L(a) = \{3\}$ for every $a \in A' \setminus N(v)$ and $L(b) = \{2\}$ for every $b \in B' \setminus N(v)$. Consequently $|L(w)| \leq 2$ for every $w \in Y$. Now the proof is complete due to Lemma 13.

So, we may assume that $N(v) \cap X \subseteq A'$. Let D be the component of the set $\{y \in Y : N(y) \cap X \subseteq A'\}$ such that $v \in D$. By Claim 15, D is complete to $N(v) \cap X$ and we may assume that D is anticomplete to $V(G) \setminus (D \cup N(v))$. We may assume that $|V(G)| \geq 7$, and thus $G[D]$ is bipartite. Let $U = N(v) \cap X$ and $W = N(v) \cap D$. Let $P = p_1\text{-}p_2\text{-}p_3\text{-}p_4$ be a path as given by Claim 11. If $p_1, p_4 \in W$, then $p_2, p_3 \in D$, contrary to the fact that D is bipartite. Therefore $p_1, p_4 \in U$, and consequently $p_2, p_3 \notin U \cup D$. Pick any $w \in W$. Now completing P via v we get a C_5, and w is adjacent to v and its two neighbors in this C_5, so the proof is complete by Claim 16. This proves (13).

We may assume that for every $y \in Y'$ there is an index i such that y is complete to C_i. (14)

Let $y \in Y'$, and recall that y is critical. First we show that y has a neighbor in A and a neighbor in B. Suppose $N(y) \cap B = \emptyset$. Then, by (13), $N(y) \subseteq A$. But now a coloring of $G \setminus y$ can be extended to a coloring of G by assigning color 2 to y, a contradiction. This proves that y has a neighbor in A and a neighbor in B.

By (12) we may assume that $N(y) \subseteq A'' \cup B''$, for otherwise (14) holds. Let $U = N(y) \cap A''$ and $W = N(y) \cap B''$; then U is complete to W. Let $P = p_1\text{-}p_2\text{-}p_3\text{-}p_4$ be a path as in Claim 11. We may assume that $p_1, p_4 \in U$. By Claim 13.3, not both of p_2 and p_3 are contained in Y. We may assume $p_2 \in X$ and thus $p_2 \notin A$ since A is stable. Since A'' is complete to B', $p_2 \notin B'$ and thus $|L(p_2)| = 1$. Since p_2 is adjacent to $p_1 \in N(y)$, $L(p_2) = \{2\}$ and p_2 is anticomplete to W since we have updated. By Claim 16 we may assume that W is complete to p_3, and by the previous argument applied to p_3 it follows that $p_3 \in Y$. Note that p_2 is anticomplete to $\{c_2, c_4\}$ since $L(c_2) = L(c_4) = \{2\}$. If p_2 is adjacent to c_1, then for every $w \in W$ the path $y\text{-}w\text{-}p_3\text{-}p_2\text{-}c_1\text{-}c_2$ is a P_6, a contradiction. Otherwise, by Claim 13.4, p_2 is adjacent to c_5, and thus $c_1\text{-}c_5\text{-}p_2\text{-}p_1\text{-}y\text{-}p_4$ is a P_6, a contradiction. This proves (14).

Let $y_1 \in Y'$ and let $C_1 \subseteq N(y_1)$. Then we may assume that no vertex of $V(G) \setminus C_1$ is mixed on A_1 (and similarly on B_1). (15)

Suppose $x \in V(G) \setminus C_1$ is mixed on A_1. Since x is mixed on C_1, and C_1 is an anticomponent of H, there exist $a_1 \in A_1$ and $b_1 \in B_1$ such that a_1b_1 is a non-edge, and x is mixed on this non-edge. Let $a_1' \in A_1$ be such that x is mixed on $\{a_1, a_1'\}$. By Lemma 7 and Lemma 8 we can precolor $T = \{x, a_1, a_1', b_1, y_1\}$ (in addition to C), and update three times with respect to T. Let Y'' be the set of vertices with lists of size 3 after updating. We claim that $Y'' = \emptyset$. Suppose not and let $v \in Y''$. By (14) there exists an index i such that v is complete to C_i. Then $i \neq 1$. Since $v \in Y''$, $\{a_1, a_1'\}$ is complete to B_i and b_1 is complete to A_i, and we have updated three times with respect to T, it follows that $L(a_1) = L(a_1') = \{3\}$ and $L(b_1) = \{2\}$. Since x has a neighbor in $\{a_1, a_1'\}$ we may assume that $L(x) \neq \{3\}$.

First consider the case that x is adjacent to a_1 and not to b_1. Then x is non-adjacent to a_1'. Choose $a_i \in A_i$. Since $x\text{-}a_1\text{-}y_1\text{-}b_1\text{-}a_i\text{-}v$ is not a P_6, it follows that x is adjacent to a_i. Since $v \in Y''$, it follows that $L(x) = \{2\}$, and therefore x is anticomplete to B_i. Choose b_i such that a_ib_i is a non-edge, then x-a_i-v-b_i-a_1'-y_1 is a P_6, a contradiction. Therefore x is adjacent to b_1 and not to a_1. Indeed, we can prove that x is complete to B_i, which is a contradiction since $L(x) \neq \{3\}$ and $v \in Y''$. This proves (15).

$$\textit{Let } v \in Y' \textit{ and let } C_i \in N(v)\textit{. Then we may assume } |A_i| = |B_i| = 1. \tag{16}$$

Suppose this is false. We may assume that $i = 1$. By (15), no vertex of $G \setminus C_1$ is mixed on A_1 and no vertex of $G \setminus C_1$ is mixed on B_1. Choose $a_1 \in A_1$ and $b_1 \in B_1$ such that a_1b_1 is an edge if possible. Then $(G \setminus (A_1 \cup B_1)) \cup \{a_1, b_1\}$ is not L-colorable, since otherwise we can color A_1 in the color of a_1 and B_1 in the color of b_1. Since we are considering a minimal list-obstruction induced by (G, L), (16) follows.

Let $Y_1 = \{y \in Y' : N(y) \subseteq (A \setminus A'') \cup (B \setminus B'')\}$, and let $Y_2 = Y' \setminus Y_1$. By (16), every $y \in Y_1$ is complete to more than one of $C_1, \ldots, C_t$, since otherwise y is not critical. We may assume that each of $C_1, \ldots, C_s$ is complete to some vertex of Y_1, and $C_{s+1} \cup \ldots \cup C_t$ is anticomplete to Y_1. Let F be the graph with vertex set $V(F) = \{1, \ldots, s\}$ where i is adjacent to j if and only if there is a vertex $y \in Y_1$ complete to $C_i \cup C_j$. We will refer to the vertices of F as $1, \ldots, s$ and $C_1, \ldots, C_s$ interchangeably.

Let $F_1, \ldots, F_k$ be the components of F, let $A(F_i) = \bigcup_{C_j \in F_i} A_j$, and let $B(F_i) = \bigcup_{C_j \in F_i} B_j$. Moreover, let $Y(F_i) = \{y \in Y_1 : N(y) \subseteq A(F_i) \cup B(F_i)\}$.

Let $i \in \{1, \ldots, k\}$ and let $T \subseteq V(G)$ be such that $A(F_i) \cup B(F_i) \cup Y_1 \subseteq T$. Then for every L-coloring of $G[T]$, both of the sets $A(F_i)$ and $B(F_i)$ are monochromatic, and the color of $A(F_i)$ is different from the color of $B(F_i)$. (17)

Let c be a coloring of $G[T]$. Let $y \in Y_1$ be such that y has a neighbor in $A(F_i)$. We may assume that y is complete to C_1, and $C_1 \in F_i$. Let $\alpha = c(A_1)$ and $\beta = c(B_1)$, where $c(A_i)$ and $c(B_i)$ denote the unique color given to the vertices in the sets A_i and B_i. Since y is complete to at least two of $C_1, \ldots, C_s$, the sets $N(y) \cap A$ and $N(y) \cap B$ are monochromatic, and $\alpha \neq \beta$. Pick any $t \in F_i$, and let P be a shortest path in F from C_1 to t. Let s be the neighbor of t in P. We may assume that $s = C_2$ and $t = C_3$. We proceed by induction and assume that $c(A_2) = \alpha$, and $c(B_2) = \beta$. Since s is adjacent to t in F, there is $y' \in Y_1$ such that y' is complete to $C_2 \cup C_3$. Then $c(y') \in \{1, 2, 3\} \setminus \{\alpha, \beta\}$. Moreover, A_2 is complete to B_3, and A_3 is complete to B_2, and so $c(A_3) \notin \{c(y'), \beta\}$ and $c(B_3) \notin \{c(y'), \alpha\}$. It follows that $c(A_3) = \alpha$ and $c(B_3) = \beta$, as required. This proves (17).

We now construct a new graph G' where we replace each F_i by a representative in A and a representative in B, as follows. Let G' be the graph obtained from $G \setminus (C_1 \cup \ldots \cup C_s \cup Y_1)$ by adding $2s$ new vertices $a_1, \ldots, a_s, b_1, \ldots, b_s$. We put

$$N_{G'}(a_i) = \{b_i\} \cup \bigcup_{a \in A(F_i)} (N_G(a) \cap V(G'))$$

and

$$N_{G'}(b_i) = \{a_i\} \cup \bigcup_{b \in B(F_i)} (N_G(b) \cap V(G')),$$

for all $i \in \{1, \ldots, s\}$. Note that, in G', the set $\{a_1, \ldots, a_s\}$ is complete to the set $\{b_1, \ldots, b_s\}$. Let $L(a_i) = \{1, 3\}$ and $L(b_i) = \{1, 2\}$ for every i. By repeated applications of Claim 8, we deduce that G' is P_6-free.

Let $A^* = (A \setminus (A'' \cup A_1 \ldots \cup A_s)) \cup \{a_1, \ldots, a_s\}$ and $B^* = (B \setminus (B'' \cup B_1 \ldots \cup B_s)) \cup \{b_1, \ldots, b_s\}$. Note that A^* is complete to B'', and B^* is complete to A''.

Let $R = G[A^* \cup B^* \cup A'' \cup B'']$.

We may assume that $|A^*| \geq 2$, and define the list systems L_1, L_2, and L_3 as follows.

$$L_1(v) = \begin{cases} \{3\} & \text{if } v \in A'' \\ \{2\} & \text{if } v \in B'' \\ L(v) & \text{if } v \notin A'' \cup B'' \end{cases}$$

$$L_2(v) = \begin{cases} \{3\} & \text{if } v \in A^* \\ L(v) & \text{if } v \notin A^* \end{cases}$$

$$L_3(v) = \begin{cases} \{2\} & \text{if } v \in B^* \\ L(v) & \text{if } v \notin B^* \end{cases}$$

Let $\mathcal{L} = \{L_1, L_2, L_3\}$. It is clear that, for every L-coloring c of G', there exists a list system $L' \in \mathcal{L}$ such that c is also an L'-coloring of G'. Recall that by (14) every vertex of Y_2 has a neighbor in A^*, a neighbor in B^*, and a neighbor in $A'' \cup B''$. Therefore, for every $L' \in \mathcal{L}$, every vertex in Y_2 is adjacent to some vertex v with $|L'(v)| = 1$. Now by Lemma 13, Lemma 7, and Lemma 8, G' contains an induced subgraph G'' such that G'' is not $L|_{V(G'')}$-colorable, and $|V(G'') \setminus V(R)| \leq 3 \cdot 36 \cdot 100$. We may assume that for any index i, $a_i \in G''$ or $b_i \in G''$, for otherwise we can just delete F_i from G contradicting the minimality of (G, L).

We claim that the subgraph induced by G on the vertex set

$$S = (V(G) \cap V(G'')) \cup Y_1 \cup \bigcup_{i=1}^{s} (A(F_i) \cup B(F_i))$$

is not $L|_S$-colorable. Suppose this is false and let c be such a coloring. By (17), for every $i \in \{1, \ldots, k\}$ the sets $A(F_i)$ and $B(F_i)$ are both monochromatic, and c can be converted to a coloring of G'' by giving a_i the unique color that appears in $A(F_i)$ and b_i the unique color that appears in $B(F_i)$, a contradiction. Thus it is sufficient to show that $|Y_1 \cup \bigcup_{i=1}^{s} (A(F_i) \cup B(F_i))|$ is bounded from above. To see this, let $T = S \setminus (A \cup B \cup Y_1)$, then $|T| < |V(G'')|$.

Next we prove that $k \leq 2^{12000}$. Partition the set of pairs $\{(a_1, b_1), \ldots, (a_s, b_s)\}$ according to the adjacency of each (a_i, b_i) in T; let $H_1, \ldots, H_l$ be the blocks of this partition. Then $l \leq 2^{12000}$. We claim that $|H_i| = 1$ for every i. Suppose for a contradiction that $(a_i, b_i), (a_j, b_j) \in H_1$. Let c be an L-coloring of $G'' \setminus \{a_i, b_i\}$. Note that, since $N(a_i) = N(a_j)$ and $N(b_i) = N(b_j)$, setting $c(a_i) = c(a_j)$ and $c(b_i) = c(b_j)$ gives an L-coloring of G'', a contradiction. This proves that $k \leq 2^{12000}$.

For each $i \in \{1, \ldots, s\}$ partition the set $\{C_j : j \in F_i\}$ according to the adjacency of C_j in T. Then $q \leq 2^{12000}$. Pick some $C_l \in C_1^i$. For each $j \in \{2, \ldots, q\}$ let Q_j be a shortest path from C_l to C_j^i in F. In G, Q_j yields a path $Q'_j = a'_1\text{-}y'_1\text{-}a'_2\text{-}y'_2\text{-}\ldots\text{-}y'_m\text{-}a'_m$ where $a'_1 \in C_l$, $a'_m \in A \cap C_j^i$, $a'_2, \ldots, a'_{m-1} \in \bigcup_{l \in \{1,\ldots,q\} \setminus \{1,j\}} A \cap C_l^i$ and $y'_1, \ldots, y'_m \in Y_1$. Let $Y(Q_j) = \{y'_1, \ldots, y'_m\}$. Since Q'_j does not contain a P_6, it follows that $|Y(Q_j)| \leq 2$. Let $Y_1^i = \bigcup_{j=2}^{q} Y(Q_j)$, and note that $|Y_1^i| \leq 2q - 2$. Moreover, let $\hat{Y} = \bigcup_{i=1}^{k} Y_1^i$, and note that $|\hat{Y}| \leq 2(q-1)k$.

Next we claim that $\hat{Y} = Y_1$. To see this, suppose that there exists a vertex $y \in Y_1 \setminus \hat{Y}$. Note that y is critical, and let c be a coloring of $G \setminus y$. We may assume that $N(y) \subseteq \bigcup_{i \in F_1} C_i$. We will construct a coloring of G'' and obtain a contradiction. By (17), for every $i \in \{2, \ldots, k\}$ both of the sets $A(F_i)$ and $B(F_i)$ are monochromatic and so we can color a_i and b_i with the corresponding colors.

Let F' be the graph with vertex set F_1 where i is adjacent to j if and only if there is a vertex $y' \in \hat{Y}$ (and therefore $y' \in Y_1^1$) complete to C_i and C_j. Recall the partition $C_1^1, \dots, C_q^1$. By the definition of Y_1^1, there exists $C_1' \in C_1^1$ such that for every $i \in \{2, \dots, q\}$ there is a path in F' from C_1' to a member C_i' of C_i^1. Write $\{a_1'\} = C_1' \cap A$ and $\{b_1'\} = C_1' \cap B$, and let $\alpha = c(a_1')$ and $\beta = c(b_i')$. Following the outline of the proof of (17) we deduce that $\alpha \neq \beta$, and that for each $i \in \{1, \dots, q\}$ some vertex of $\bigcup_{C \in C_i^1} C \cap A$ is colored in color α, and some vertex of $\bigcup_{C \in C_i^1} C \cap B$ is colored in color β. Observe that for every index i only vertices of $Y_1 \cup \bigcup_{C \in C_i^1} C \cap B$ are mixed on $\bigcup_{C \in C_i^1} C \cap A$, and only vertices of $Y_1 \cup \bigcup_{C \in C_i^1} C \cap A$ are mixed on $\bigcup_{C \in C_i^1} C \cap B$. Thus we can color a_1 with color α and b_1 with color β, obtaining a coloring of G'', a contradiction. This proves that $|Y_1| \leq 2 \cdot 2^{24000}$. Now applying Claim 10 $|Y_1|$ times implies that

$$\bigcup_{i=1}^{s} (A(F_i) \cup B(F_i)) \leq 7 \cdot 2 \cdot 2^{24000} \cdot 2^{12000} = 14 \cdot 2^{36000}.$$

Hence there exists a minimal list-obstruction induced by (G, L) which has at most $|S| \leq |Y_1| + |\bigcup_{i=1}^{s} (A(F_i) \cup B(F_i))| \leq 20 \cdot 2^{36000}$ vertices. So by Lemma 7 and Lemma 8, $|V(G)| \leq 3^5 \cdot 36 \cdot |S| \leq 10^6 \cdot 2^{36000}$. This completes the proof. □

The statement of Lemma 14 follows from Lemma 9, 12 and 17.

6 $2P_3$-free 4-vertex critical graphs

The aim of this section is to show that there are only finitely many $2P_3$-free 4-vertex critical graphs. The proof of this fact is roughly along the lines of the proof in the P_6-free list-obstruction case.

Lemma 11 shows that there are infinitely many $2P_3$-free minimal list-obstructions where every list has size at most two. This is different in the P_6-free case, see Lemma 13. However if we add the additional assumption that the minimal list-obstruction is contained in a $2P_3$-free 4-vertex-critical graph and was obtained by updating with respect vertices with a fixed color, we can show that its size is bounded.

Lemma 16. *Let (G, L) be a list obstruction. Assume that G is $2P_3$-free and the following holds.*

(a) Every list contains at most two entries.

(b) Every vertex v of G with $|L(v)| = 2$ has a neighbor u with $|L(u)| = 1$ such that for all $w \in V(G)$ with $|L(w)| = 2$, $uw \in E(G)$ implies $L(w) = L(v)$.

Then (G, L) contains a minimal list-obstruction whose order is bounded by some constant.

Like in the case of P_6-free list-obstructions, we can use the precoloring technique to see that the lemma above implies our main lemma.

Lemma 17. *There are only finitely many $2P_3$-free 4-vertex-critical graphs.*

6.1 Proof of Lemma 16

Let G' be an induced subgraph of G such that $(G', L|_{G'})$ is a minimal list-obstruction. According to Lemma 12, it suffices to prove that the length of any propagation path of $(G', L_{G'})$ is bounded by a constant. To see this, let $P = v_1\text{-}v_2\text{-}\dots\text{-}v_n$ be a propagation path of $(G', L_{G'})$ starting with color α, say. Consider v_1 to be colored with α, and update along P until every vertex of P is colored.

Let this coloring of P be denoted c. Recall condition (1) from the definition of propagation path: every edge v_iv_j with $3 \leq i < j \leq n$ and $i \leq j-2$ is such that

$$S(v_i) = \alpha\beta \text{ and } S(v_j) = \beta\gamma,$$

where $\{1,2,3\} = \{\alpha, \beta, \gamma\}$.

First we prove that there is a constant δ such that there is a subpath $Q = v_m\text{-}v_{m+1}\text{-}\ldots\text{-}v_{m'}$ of P of length at least $\lfloor \delta n \rfloor$ with the following property. After permuting colors if necessary, it holds for all $i \in \{m, \ldots, m'\}$ that

$$S(v_i) = \begin{cases} 32, & \text{if } i \equiv 0 \mod 3 \\ 13, & \text{if } i \equiv 1 \mod 3 \\ 21, & \text{if } i \equiv 2 \mod 3 \end{cases}.$$

To see this, suppose there are two indices $i, j \in \{3, \ldots, n-3\}$ such that $i+2 \leq j$ and $c(v_i) = c(v_{i+2}) = c(v_j) = c(v_{j+2})$. Moreover, suppose that $c(v_i) = c(v_{i+2}) = c(v_j) = c(v_{j+2}) = \alpha$ and $c(v_{i+1}) = c(v_{j+1}) = \beta$ for some α, β with $\{\alpha, \beta, \gamma\} = \{1,2,3\}$. Thus, $L(v_{i+1}) = L(v_{i+2}) = L(v_{j+1}) = L(v_{j+2}) = \{\alpha, \beta\}$, $\alpha \in L(v_{j+3})$, and $\alpha \neq c(v_{j+3})$. But now $v_i\text{-}v_{i+1}\text{-}v_{i+2}$ and $v_{j+1}\text{-}v_{j+2}\text{-}v_{j+3}$ are both induced P_3's, according to (1), and there cannot be any edge between them. This is a contradiction to the assumption that G is $2P_3$-free. The same conclusion holds if $c(v_{i+1}) = c(v_{j+1}) = \gamma$. Hence, there cannot be three indices $i, j, k \in \{3, \ldots, n-3\}$ such that $i+2 \leq j$, $j+2 \leq k$, and

$$c(v_i) = c(v_{i+2}) = c(v_j) = c(v_{j+2}) = c(v_k) = c(v_{k+2}) = \alpha.$$

Consider the following procedure. Pick the smallest index $i \in \{3, \ldots, n-3\}$ such that $c(v_i) = c(v_{i+2}) = 1$, if possible, and remove the vertices v_i, v_{i+1}, and v_{i+2} from P. Let P' be the longer of the two paths $v_1\text{-}v_2\text{-}\ldots\text{-}v_{i-1}$ and $v_{i+3}\text{-}v_2\text{-}\ldots\text{-}v_n$. Repeat the deletion process and let $P'' = v_r\text{-}v_{r+1}\text{-}\ldots\text{-}v_{r'}$ be the path obtained. As shown above, we now know that there is no index $j \in \{r+2, \ldots, r'-3\}$ with $c(v_j) = c(v_{j+2}) = 1$.

Repeating this process for colors 2 and 3 shows that there is some $\delta > 0$ such that there is a path $Q = v_m\text{-}v_{m+1}\text{-}\ldots\text{-}v_{m'}$ of length $\lfloor \delta n \rfloor$ where $c(v_i) \neq c(v_{i+2})$ for all $i \in \{m-1, \ldots, m'-2\}$. Thus, after swapping colors if necessary we have the desired property defined above.

From now on we assume that G has sufficiently many vertices and hence $m' - m$ is sufficiently large. Since G is $2P_3$-free and hence P_7-free, the diameter of every connected induced subgraph of G is bounded by a constant. In particular, the diameter of the graph $G[\{v_m, \ldots, v_{m'}\}]$ is bounded, and so we may assume that there is a vertex v_i with $m \leq i \leq m'$ with at least 20 neighbors in the path Q. We may assume that $c(v_i) = 1$ and, thus, $S(v_i) = 13$.

We discuss the case when $|N(v_i) \cap \{v_m, \ldots, v_{i-1}\}| \geq 10$. The case of $|N(v_i) \cap \{v_{i+1}, \ldots, v_{m'}\}| \geq 10$ can be dealt with in complete analogy.

We pick distinct vertices $v_{i_1}, \ldots, v_{i_{10}} \in N(v_i) \cap \{v_m, \ldots, v_{i-1}\}$ where $i_1 < i_2 < \ldots < i_{10}$. Note that (1) implies that $S(v_{i_j}) = 21$ for all $j \in \{1, \ldots, 10\}$.

We can pick three indices j_1, j_2, j_3 with $r' < j_1 < j_2 < j_3 < m'$ such that

- $S(v_{j_1}) = S(v_{j_2}) = S(v_{j_3}) = 32$, and
- $i_2 + 5 = j_1$, $j_1 + 6 = j_2$, $j_2 + 4 \leq i_7$, $i_8 + 5 = j_3$, and $j_3 + 4 = i$.

Recall that assumption (b) of the lemma we are proving implies the following. Since $L(v_{j_u}) = \{2,3\}$, v_{j_u} has a neighbor x_{j_u} with $L(x_{j_u}) = \{1\}$, $u = 1,2,3$, such that x_{j_u} is not adjacent to any vertex v_j with $m \leq j \leq m'$ and $j \equiv 1 \mod 3$ or $j \equiv 2 \mod 3$.

Suppose that $x_{j_u} = x_{j_{u'}}$ for some $v_{j_{u'}}$ with $u' \in \{1,2,3\} \setminus \{u\}$. Now the path v_{j_u}-x_{j_u}-$v_{j_{u'}}$ is an induced P_3, and so is the path v_{i_1}-v_i-v_{i_2}, both according to condition (1). Moreover, there is no edge between those two paths, due to (1), which is a contradiction. Hence, the three vertices x_{j_u}, x_{j_u}, and x_{j_u} are mutually distinct and, due to the minimality of (G, L), mutually non-adjacent.

Consider the induced P_3's v_{j_1+1}-v_{j_1}-x_{j_1} and v_{j_3+1}-v_{j_3}-x_{j_3}. Since G is $2P_3$-free, there must be an edge between these two paths. According to (1), it must be the edge $v_{j_1+1}v_{j_3}$. For similar reasons, the edge $v_{j_2+1}v_{j_3}$ must be present. Now the path v_{j_1+1}-v_{j_3}-v_{j_2+1} is an induced P_3, and so is the path v_{i_7}-v_i-v_{i_8}. Moreover, there is no edge between those two paths, due to (1), which is a contradiction. This completes the proof.

6.2 Proof of Lemma 17

We start with two statements that allow us to precolor sets of vertices with certain properties.

Claim 18. *Assume that (G, L) is a list-obstruction. Let $X \subseteq V(G)$ be such that there exists a coloring c of $G[X]$ with the following property: for each $x \in X$ there exists a set $N_x \subseteq V(G)$ with $|N_x| \leq k$ such that x is colored $c(x)$ in every coloring of $(G[\{x\} \cup N_x], L|_{\{x\} \cup N_x})$. Let L' be a list system such that*

$$L'(v) = \begin{cases} L(v), & \text{if } v \in V(G) \setminus X \\ \{c(x)\}, & \text{if } v \in X \end{cases}.$$

Then the following holds.

(a) (G, L') is a list-obstruction.

(b) If $K \subseteq V(G)$ is such that $(G[K], L'|_K)$ is a minimal list-obstruction induced by (G, L'), then (G, L) contains a minimal list-obstruction of size at most $(k+1)|K|$.

Proof. Since $L'(v) \subseteq L(v)$ for all $v \in V(G)$, (G, L') is also a list-obstruction. This proves (a).

Let $A = G[K \cup \bigcup_{x \in K \cap X} N_x]$, then $|V(A)| \leq (k+1)|K|$. Suppose that there exists a coloring, c' of $(A, L|_A)$. Note that for every $x \in V(A)$, $N_x \subseteq A$. Hence by the definition of X, $c'(x) = c(x)$ for every $x \in V(A)$. This implies that c' is also a coloring of $(A, L'|_A)$, which gives a coloring of $(G[K], L'|_K)$, a contradiction. Therefore $(A, L|_A)$ is a list-obstruction induced by (G, L). Since $|V(A)| \leq (k+1)|K|$, (b) holds. This completes the proof. □

Claim 19. *Let (G, L) be a list-obstruction, and let $X \subseteq V(G)$ be a vertex subset such that $|L(x)| = 1$ for every $x \in X$. Let $Y = N(X)$, and let $Y' \subseteq Y$ be such that for every $v \in Y'$, $|L(v)| = 3$. For every $v \in Y'$, pick $x_v \in N(v) \cap X$. Let L' be the list defined as follows.*

$$L'(v) = \begin{cases} L(v), & \text{if } v \in V(G) \setminus Y' \\ L(v) \setminus L(x_v), & \text{if } v \in Y' \end{cases}.$$

Let $(G', L'|_{G'})$ be a minimal list-obstruction induced by (G, L'). Then there exists a minimal list-obstruction induced by (G, L), say $(G'', L|_{G''})$, with $|V(G'')| \leq 2|V(G')|$.

Proof. Let $R = \{x_v : v \in V(G') \cap Y'\}$ and let $P = R \cup V(G')$. It follows that $|V(P)| \leq 2|V(G')|$. It remains to prove that $(G[P], L|_{G[P]})$ is not colorable. Suppose there exists a coloring c of $(G[P], L|_{G[P]})$. Note that c is not a coloring of $(G', L'|_{G'})$ and G' is an induced subgraph of $G[P]$. Hence there exists $w \in V(G')$ such that $c(w) \notin L'(w)$. By the construction of L', it follows that $w \in Y'$ and that $c(w) \in L(w) \setminus L'(w) = \{c(x_w)\}$, which is a contradiction. This completes the proof. □

Let G be a $2P_3$-free 4-vertex-critical graphs such that $|V(G)| \geq 5$, then the following claim holds.

Claim 20. *At least one of the following holds*

1. *There exists $S_0 \subseteq V(G)$ such that $|S_0| \leq 5$, $G[S_0]$ contains a copy of P_3 and $S_0 \cup N(S_0) \cup N(N(S_0)) = V(G)$, or*

2. *G has a semi-dominating set of size at most 5.*

Proof. Since G is $2P_3$-free and thus also P_7-free, Theorem 15 states that G has a dominating induced P_5 or a dominating P_5-free connected induced subgraph, denoted by D_f. Recall that a dominating set is always a semi-dominating set; so we may assume that the latter case holds and $|V(D_f)| \geq 6$. By applying Theorem 15 to D_f again, we deduce that D_f has a dominating induced subgraph T, which is isomorphic to P_3 or a connected P_3-free graph.

If T is isomorphic to P_3, then we are done by setting $S_0 = V(T)$. Hence we may assume T is a connected P_3-free graph. Therefore T is a complete graph, and so $V(T) \leq 3$. If there exists a vertex $s' \in V(G \setminus T)$ mixed on T, we are done by setting $S_0 = V(T) \cup \{s'\}$. Hence we may assume that for every $v \in V(D_f \setminus T)$, v is complete to T. Since $|V(G)| \geq 5$, it follows that D_f is K_4-free. Therefore there exist $v, w \in V(D_f \setminus T)$ such that v is non-adjacent to w and we are done by setting $S_0 = V(T) \cup \{v, w\}$. □

If G has a semi-dominating set of size at most 5, we are done by Lemma 9 and Lemma 16. Hence we may assume there exists S_0 defined as in Claim 20.

For a list system L' of G, we say that (X_1, X_2, B, S) is the *partition with respect to* L' by setting:

(a) $S = \{v \in V(G) : |L'(v)| = 1\}$.

(b) $B = N(S)$, by Claim 19 we may assume that $|L'(v)| = 2$ for every $v \in B$.

(c) Let $X = V(G) \setminus (S \cup B)$. We say that C is a *good component* of X if there exist $x \in C$ and $\{i, j\} \subseteq \{1, 2, 3\}$ so that x has two adjacent neighbors $a, b \in B_{ij}$, where $B_{ij} = \{b \in B$ such that $L'(b) = \{i, j\}\}$. Let X_1 be the union of all good components of X and let $X_2 = X \setminus X_1$.

Let (X_1, X_2, B, S) be the partition with respect to L'. Define $X = X_1 \cup X_2$. For every $1 \leq i \leq j \leq 3$, define $B_{ij} = \{b \in B$ such that $L'(b) = \{i, j\}\}$ and $X_{ij} = \{x \in X_2$ such that $|N(x) \cup B_{ij}| \geq 2\}$. For $\{i, j, k\} = \{1, 2, 3\}$, let us say that a component C of X_2 is *i-wide* if there exist a_j in B_{ik} and a_k in B_{ij} such that C is complete to $\{a_j, a_k\}$. We call a_j and a_k *i-anchors* of C. Note that a component can be i-wide for several values of i. Let L'' be a subsystem of L' and let (X'_1, X'_2, B', S') be the partition with respect to L''. Then $S \subseteq S'$, $B' \setminus B \subseteq X_1 \cup X_2$ and $X'_2 \subseteq X_2$.

Next we define a sequence of new lists $L_0, .., L_5$. Let $\{i, j, k\} = \{1, 2, 3\}$.

1. Let L_1 be the list system obtained by precoloring S_1 and updating three times. Let (X^1_1, X^1_2, B^1, S^1) be the partition with respect to L_1.

2. For each $k \in \{1, 2, 3\}$, choose $x_k \in X^1_{ij}$ such that $|N(x_k) \cap B^1_{ij}|$ is minimum. Let $a_k, b_k \in N(x_k) \cap B^1_{ij}$. Let L_2 be the list system obtained from L_1 by precoloring $\bigcup_{i=1}^{3} \{a_i, b_i, x_i\}$ and updating the lists of vertices three times. Let (X^2_1, X^2_2, B^2, S^2) be the partition with respect to L_2 .

3. For each $k \in \{1,2,3\}$, let $\hat{B}_k \subseteq B^2_{ij}$ with $|\hat{B}_k| \leq 1$ be defined as follows. If there does not exist a vertex v that starts a path $v-u-w$ where $u,w \in X^2_2$, then $\hat{B}_k = \emptyset$. Otherwise choose $b_k \in B^2_{ij}$ maximizing the number of pairs (u,w) where $v-u-w$ is a path and let $\hat{B}_k = \{b_k\}$. Let L_3 be the list system from L_2 obtained by precoloring $\hat{B}_1 \cup \hat{B}_2 \cup \hat{B}_3$ and updating three times. Let (X^3_1, X^3_2, B^3, S^3) be the partition with respect to L_3.
4. Apply step 2 to (X^3_1, X^3_2, B^3, S^3) with list system L_3; let L_4 be the list system obtained and let (X^4_1, X^4_2, B^4, S^4) be the partition with respect to L_4.
5. For every component C_t of X^4_2 with size 2, if C_t is i-wide with i-anchors a^t, b^t, set $L_5(a^t) = L_5(b^t) = \{i\}$; then let L_5 be the list system after updating with respect to $\bigcup_t \{a^t, b^t\}$ three times. Let (X^5_1, X^5_2, B^5, S^5) be a partition with respect to L_5.

By Lemma 7 and Lemma 8, it is enough to prove that (G, L_4) induces a bounded size list obstruction. We plan to use Claim 18 and Lemma 8 to prove the same for (G, L_5), but first we need a few technical statements.

Claim 21. *Let $1 \leq m \leq l \leq 5$. Then the following hold.*

1. *For every vertex in $x \in X^m$, $|L_m(x)| = 3$, and every component of X^m is a clique with size at most 3.*
2. *If no vertex of B^m_{ij} is mixed on an edge in $G[X^m_2]$, then no vertex of B^l_{ij} is mixed on an edge in $G[X^l_2]$.*
3. *If no vertex of X^m_2 has two neighbors in B^m_{ij} and no vertex of B^m is mixed on an edge in $G[X^m_2]$, then no vertex of X^l_2 has two neighbors in B^l_{ij}.*

Proof. By construction, for every vertex in $x \in X^m$, $|L_m(x)| = 3$. Observe that $S_0 \subseteq S^m$. Recall that G is $2P_3$-free and that S_0 contains a P_3. Hence X^m does not contain a P_3, and so every component of X^m is a clique. Since $|V(G)| \geq 5$, it follows that every component of X^m has size at most 3. This proves the first statement.

Let $b \in B^l_{ij}$ be mixed on the edge uv such that $u,v \in X^l_2$. Recall that $X^l_2 \subseteq X^m_2$; thus $\{u,v\} \subseteq X^m_2$. By assumption $b \notin B^m_{ij}$ and hence $b \in X^m$. But now $b-u-v$ is a P_3 in X^m, a contradiction. This proves the second statement.

To prove the last statement, suppose that there exists $y \in X^l_2$ with two neighbors $u,v \in B^l_{23}$. Since $y \in X^l_2$, it follows that u,v are non-adjacent. Note that $y \in X^m_2$, hence by assumption and symmetry, we may assume that $v \notin B^m$. Therefore $v \in X^m$. Since X^m_1 is the union of components of X^m, and $y \in X^m_2$ is adjacent to v, it follows that $v \in X^m \setminus X^m_1$, and consequently $v \in X^m_2$. If $u \notin B^m$, then u-y-v is a P_3 in $G[X^m]$, contrary to the first statement. Hence $u \in B^m$ and then u is mixed on the edge vy of $G[X^m_2]$, a contradiction. This completes the proof. □

Claim 22. $X^1_{12} \cup X^1_{23} \cup X^1_{13} \subseteq B^2 \cup S^2$.

Proof. Suppose that there exists $x' \in X^1_{ij} \setminus (B^2 \cup S^2)$ for some $1 \leq i \leq j \leq 3$; then $|L_2(x')| = 3$. Let $x_k \in X^1_{ij}$ and $a_k, b_k \in N(x_k) \cap B^1_{ij}$ be the vertices chosen to be precolored in the step creating L_1. Then x' is non-adjacent to $\{x_k, a_k, b_k\}$. The minimality of $|N(x_k) \cap B^1_{ij}|$ implies that there exist $a', b' \in (N(x') \cup B^1_{ij}) \setminus N(x_k)$. Since G is $2P_3$-free, there exists an edge between $\{a_k, b_k, x_k\}$ and $\{a', b', x'\}$. Specifically, there exists an edge between $\{a_k, b_k\}$ and $\{a', b'\}$. We may assume that $L_2(a_k) = \{i\}$ and a_k is adjacent to at least one of a', b'. Recall that L_1 is obtained by precoloring $\bigcup_{i=1}^{3} \{a_i, b_i, x_i\}$ and updating three times. It follows that $j \notin L_2(x')$, a contradiction. □

Claim 23. *No vertex of B^3 is mixed on an edge of X_2.*

Proof. Suppose that there exists a path b'-x_1'-x_2' such that $b' \in B^3_{ij}$ and $x_1', x_2' \in X^3_2$. Note that $x_1', x_2' \in X^2_2$ since L_3 is a subsystem of L_2. By Claim 21.1, X^2 is P_3-free. Hence $b' \in B^2_{ij}$. By Claim 21.3, there exists $b \in B^2_{ij}$ such that $b - x - y$ is a path where $x, y \in X^2_2$. Then in step 3, $\hat{B}_k \neq \emptyset$ and let $b \in \hat{B}_k$. By the construction of L_3 and since $x_1', x_2' \in X^3_2$, b is anticomplete to $\{b', x_1', x_2'\}$. By the construction of $\hat{B}_k$, there exist $x_1, x_2 \in X^2_2$ such that b-x_1-x_2 is a path and b' is not mixed on x_1x_2. If $\{x_1, x_2\}$ is not anticomplete to $\{x_1', x_2'\}$, then by Claim 21.1 $G[\{x_1, x_2, x_1', x_2'\}]$ is a K_4, a contradiction to the fact that $|V(G)| \geq 5$. Hence $\{x_1, x_2\}$ is anticomplete to $\{x_1', x_2'\}$. Since G is $2P_3$-free, there exists an edge between b' and $\{x_1, x_2\}$. Consequently, b' is complete to $\{x_1, x_2\}$. Now x_1 has two neighbors in B^2_{ij}, namely b and b'. By Claim 22, $x_1 \notin B^1_{ij}$. It follows that either $b \in X^1$ or $b' \in X^1$. If $b \in X^1$, then $b - x - y$ is a P_3 in X^1, contrary to Claim 21.1. Hence $b' \in X^1$. It follows that b'-x_1'-x_2' is a P_3 in X^1, again contrary to Claim 21.1. This completes the proof. □

By Claim 21 and Claim 23, no vertex of B^4 is mixed on an edge of $G[X^4_2]$. Consider a component $C_t = \{x, y\}$ with i-anchors a_t, b_t chosen in step 5, then $\{xy, a_tx, a_ty, b_tx, b_ty\} \subseteq E(G[\{x, y, a_t, b_t\}])$. By the definition of i-anchors, $L_4(a_t) \cap L_4(b_t) = \{i\}$. Then it is clear that $c(a_t) = c(b_t) = i$ for every coloring c of $(G[\{x, y, a_t, b_t\}], L_4|_{G[\{x,y,a_t,b_t\}]})$. Hence we can applying Claim 18 on L_4. By Claim 18 and Lemma 8, it is enough to show that (G, L_5) induces a bounded size list obstruction.

Claim 24. *X^5_2 is stable.*

Proof. Since $|V(G)| \geq 5$ and since no vertex of B^5 is mixed on an edge of $G[X^5_2]$, by Claim 21.1 every component of X^5_2 has size at most 2. We may assume some component C of X^5_2 has size exactly 2, for otherwise the claim holds. Then C is a component of X^4_2. By Claim 21 and Claim 22, no vertex of X^4_2 has two neighbors in B^4_{ij}. Since every vertex in G has degree at least 3, every vertex of C has a neighbor in at least two of $B^4_{12}, B^4_{23}, B^4_{13}$. It follows that C is i-wide for some i and therefore $C \subseteq S^5 \cup B^5$, a contradiction. □

By Claim 21 and Claim 22, no vertex of X^5_2 has two neighbors in B^5_{ij}. Since every vertex in G has degree at least 3, it follows that every vertex of X^5_2 has exactly one neighbor in each of B^5_{ij}. Let Y_0, Y_1, …, Y_6 be a partition of X^5_2 as follows. Let $x \in X^5_2$ and $a_k = N(x) \cap B^5_{ij}$ for $\{i, j, k\} = \{1, 2, 3\}$. If $\{a_1, a_2, a_3\}$ is a stable set, then $x \in Y_0$; if $E(G[\{a_1, a_2, a_3\}]) = \{a_ia_j\}$, then $x \in Y_k$; and if $E(G[\{a_1, a_2, a_3\}]) = \{a_ia_j, a_ia_k\}$, then $x \in Y_i$. Note that $G[\{a_1, a_2, a_3\}]$ cannot be a clique since $V(G) \geq 5$. For each non-empty Y_s, pick $x_s \in Y_s$, and let $a_{sk} \in N(x_s) \cap B^5_{ij}$ for $\{i, j, k\} = \{1, 2, 3\}$. Let L_6 be the list system obtained by precoloring $\bigcup_{i=0}^{6} \{x_i, a_{i1}, a_{i2}, a_{i3}\}$ with c and updating three times.

Claim 25. *For every $x \in X^5_2$, $|L_6(x)| \leq 2$*

Proof. Suppose there exists $y \in Y_i$ such that $|L_6(y)| = 3$; let $b_1 = N(y) \cap B^5_{23}$, $b_2 = N(y) \cap B^5_{13}$, and $b_3 = N(y) \cap B^5_{12}$. Then $\{a_{i1}, a_{i2}, a_{i3}, x_i\}$ and $\{b_1, b_2, b_3, y\}$ are disjoint sets. Note that $c(a_{i1}), c(a_{i2}), c(a_{i3})$ can not all be pairwise different, at so by symmetry we may assume that $c(a_{i1}) = c(a_{i2}) = 3$ and $c(a_{i3}) = 2$. Thus, $a_{i1}a_{i2}$ is a non-edge. By the construction of L_6 and since $|L_6(y)| = 3$, the only possible edges between the sets $\{a_{i1}, a_{i2}, a_{i3}\}$ and $\{b_1, b_2, b_3\}$ are $a_{i3}b_2$, $a_{i1}b_3$ and $a_{i2}b_3$. Recall that every vertex of X^5_2 has exactly three neighbors in B^5, and so $N(y) \cap B^5 = \{b_1, b_2, b_3\}$ and $N(x_i) \cap B^5 = \{a_{i1}, a_{i2}, a_{i3}\}$. Since $G[\{a_{i1}, x, a_{i2}, b_1, y, b_2\}]$ is not a $2P_3$, it follows that b_1 is adjacent to b_2. But this contradicts to the fact that both x_i and y belong to Y_i. □

Let (X_6^2, X_6^2, B^6, S^6) be the partition with respect to L_2. For every component $C_s \subseteq X_1^6$, let $x_s^k \in C_s$ be such that x has two adjacent neighbors in B_{ij}^6 and set $L^*(x) = \{k\}$; let P be the set of all such vertices and let L^* be the list system after updating with respect to P three times. Pick $x \in P$, we may assume that there exist $a, b \in N(x) \cap B_{ij}^6$. Then $L_6(a) = L_6(b) = \{i, j\}$. As a result, for every coloring c of $(G[\{x, a, b\}], L_6|_{\{}x, a, b\})$, $c(x) = k$. This implies that we can apply Claim 18 to L_6. By Lemma 7, Lemma 8 and Claim 18, it is enough to prove that (G, L^*) induces a bounded size list obstruction. Let (X_1^*, X_2^*, B^*, S^*) be the partition with respect to L^*. Then X_1^*, X_2^* are empty. Now (G, L^*) satisfies the hypotheses of Lemma 16, and this finishes the proof of Lemma 17.

7 $P_4 + kP_1$-free minimal list-obstructions

In this section we prove that there are only finitely many $P_4 + kP_1$-free list-obstructions. This also implies that there are only finitely many $P_4 + kP_1$-free 4-vertex-critical graphs.

Lemma 18. *Let (G, L) be a minimal obstruction such that each list has at most two entries. Moreover, let G be $(P_4 + kP_1)$-free, for some $k \in \mathbb{N}$. Then $V(G)$ is bounded from above by a constant depending only on k.*

Proof. According to Lemma 12, it suffices to prove that every propagation path in (G, L) has a bounded number of vertices. To see this, let $P = v_1\text{-}\ldots\text{-}v_n$ be a propagation path in (G, L) starting with color α, say. Consider v_1 to be colored with α, and update along P until every vertex is colored. Call this coloring c. Suppose that $n \geq 100k^2 + 100$. Our aim is to show that this assumption is contradictory. Recall condition (1) from the definition of propagation path: every edge v_iv_j with $3 \leq i < j \leq n$ and $i \leq j - 2$ is such that

$$S(v_i) = \alpha\beta \text{ and } S(v_j) = \beta\gamma,$$

where $\{1, 2, 3\} = \{\alpha, \beta, \gamma\}$.

First we suppose that there is a sequence $v_i, v_{i+1}, \ldots, v_j$ with $2 \leq i \leq j \leq n$ and $j - i \geq 5 + 2k$ such that $c(v_{i'}) = c(v_{i'+2})$ for all i' with $i \leq i' \leq j - 2$. But then (1) implies that $v_{i+1}\text{-}v_{i+2}\text{-}\ldots\text{-}v_j$ is an induced path, and thus G is not $P_4 + kP_1$-free, a contradiction.

Suppose now that there is an index i with $2 \leq i \leq \lceil n/2 \rceil - 3$ such that $c(v_i) = c(v_{i+2}) = \alpha$ and $c(v_{i+1}) = c(v_{i+3}) = \beta$. In particular, $L(v_{i+3}) = \{\alpha, \beta\}$. Now condition (1) of the definition of a propagation path implies that there cannot be an edge between v_i and v_{i+3}, and so $v_i\text{-}v_{i+1}\text{-}v_{i+2}\text{-}v_{i+3}$ is an induced P_4. Therefore no such sequence exists.

We now pick k disjoint intervals of the form $\{j, \ldots, j + 7 + 2k\} \subseteq \{\lceil n/2 \rceil + 1, \ldots, n\}$. As shown above, each of these intervals contains an index i' in its interior with $c(v_{i'}) = \alpha$. These $v_{i'}$ form a stable set and (1) implies that the induced path $v_i\text{-}v_{i+1}\text{-}v_{i+2}\text{-}v_{i+3}$ is anticomplete to each $v_{i'}$, a contradiction to the fact that G is $P_4 + kP_1$-free.

Now suppose that there is an index i with $r+1 \leq i \leq \lceil (r+s)/2 \rceil - 3$ such that $c(v_i) = c(v_{i+2}) = \alpha$ and $c(v_{i+1}) = \beta$. From what we have shown above we know that $c(v_{i-1}) = c(v_{i+3}) = \gamma$, where $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$. Thus, we have $S(v_i) = \alpha\gamma$, $S(v_{i+1}) = \beta\alpha$, $S(v_{i+2}) = \alpha\beta$, and $S(v_{i+3}) = \gamma\alpha$. According to (1), the path $v_i\text{-}v_{i+1}\text{-}v_{i+2}\text{-}v_{i+3}$ is induced.

Pick a vertex v_j with $\lceil n/2 \rceil + 1 \leq j \leq n$. According to (1), v_j is anticomplete to the path $v_i\text{-}v_{i+1}\text{-}v_{i+2}\text{-}v_{i+3}$ unless one of the following holds.

(a) $S(v_j) = \alpha\beta$,

(b) $S(v_j) = \alpha\gamma$,

(c) $S(v_j) = \beta\gamma$, or

(d) $S(v_j) = \gamma\beta$.

Let us say that v_j is of *type A* if it satisfies one of the above conditions. If v_j is not of type A, we say it is of *type B*.

We claim that there are at most $3k - 3$ vertices of type B. To see this, suppose there are at least $3k - 2$ vertices of type B. By definition, each vertex of type B is anticomplete to the set the path v_i-v_{i+1}-v_{i+2}-v_{i+3}. Since (G, L) is a minimal obstruction and not every vertex is of type B, the graph induced by the vertices of type B is 3-colorable. Picking the vertices of the majority color yields a set S of k independent vertices of type B. But now the set $\{v_i, \ldots, v_{i+3}\} \cup S$ induces a $P_4 + kP_1$ in G, a contradiction.

So, there are at most $3k - 3$ vertices of type B. Suppose there are more than $(3k - 2)(7 + 2k)$ many vertices of type A. Then there is an index $t \geq \lceil n/2 \rceil + 1$ such that $v_t + j'$ is of type A for all $j' \in \{0, \ldots, 6+2k\}$. Suppose that there is an index $j' \in \{0, \ldots, 5+2k\}$ such that $c(v_{t+j'}) = \alpha$. Then $S(v_{t+j'+1}) = \cdot\, \alpha$, in contradiction to the fact $v_{t+j'+1}$ is of Type A. So, for all $j' \in \{0, \ldots, 5+2k\}$ we have that $c(v_{t+j'}) \neq \alpha$, in contradiction to what we have shown above. Summing up, n is bounded by $2(3k - 2)(7 + 2k) + 1$ if there is an index i with $2 \leq i \leq \lceil n/2 \rceil - 3$ such that $c(v_i) = c(v_{i+2}) = \alpha$ and $c(v_{i+1}) = \beta$.

Hence, our assumption $n \geq 100k^2 + 100$ implies that $c(v_i) \neq c(v_{i+2})$ for all i with $2 \leq i \leq \lceil n/2 \rceil - 3$. This means that, without loss of generality,

$$c(v_i) = \begin{cases} 1, & i = 1\ (3) \\ 2, & i = 2\ (3) \\ 3, & i = 0\ (3) \end{cases} \tag{18}$$

for all i with $2 \leq i \leq \lceil n/2 \rceil - 3$.

Consider the path v_4-v_5-$\ldots$-v_{7+2k}. Since G is $P_4 + kP_1$-free, this is not an induced path. Hence, there is an edge of the form v_iv_j with $i < j$. If $S(v_i) = \alpha\beta$, we must have $S(v_j) = \beta\gamma$, due to (1). Consequently, $S(v_{i-1}) = \beta\gamma$, and $S(v_{j+1}) = \alpha\beta$. In particular, (1) implies that v_{i-1} is non-adjacent to v_{j+1}, and so v_{i-1}-v_i-v_j-v_{j+1} is an induced path.

Like above, we now pick k disjoint intervals of the form $\{j, \ldots, j+7+2k\} \subseteq \{\lceil n/2 \rceil + 1, \ldots, n\}$. Each of these intervals contains an index i' in it's interior with $c(v_{i'}) = \alpha$. These $v_{i'}$ form a stable set and (1) implies that the induced path v_{i-1}-v_i-v_j-v_{j+1} is anticomplete to each $v_{i'}$, a contradiction. This completes the proof. □

Using the above statement, we can now derive our main lemma.

Lemma 19. *There are only finitely many $P_4 + kP_1$-free list-obstructions, for all $k \in \mathbb{N}$.*

Proof. Let (G, L) be a $P_4 + kP_1$-free list obstruction. If G is P_4-free, we are done, since there is only a finite number of P_6-free minimal obstructions. So, we may assume that G contains an induced P_4, say v_1-v_2-v_3-v_4. Let $R = V(G) \setminus N(\{v_1, v_2, v_3, v_4\})$. Let S be a maximal stable set in R; then every vertex of $V(R) \setminus S$ has a neighbor in S. Since G is $P_4 + kP_1$-free, it follows that $|S| \leq k - 1$, and so $\{v_1, v_2, v_3, v_4\} \cup S$ is a dominating set of size at most $k + 3$ in G. Now Lemma 19 follows from Lemma 18 and Claim 9. □

8 Proof of Theorem 2 and Theorem 3

We now prove our main results, starting with the 4-vertex-critical version. First we prove Theorem 2, which we restate:

Theorem 2. *Let H be a graph. There are only finitely many H-free 4-vertex critical graphs if and only if H is a subgraph of P_6, $2P_3$, or $P_4 + kP_1$ for some $k \in \mathbb{N}$.*

Proof. If H contains a cycle or a claw, there is an infinite list of obstructions by Lemma 10. Thus, we may assume that H is a linear forest. Let $H_1, H_2, \dots H_k$ be the components of H, ordered such that $|H_1| \geq |H_2| \geq \dots \geq H_k$.

Assume that H contains an induced $2P_3$. If $H = 2P_3$, then there is a finite list of H-free 4-vertex-critical graphs by Lemma 17. So, we may assume that H contains $2P_3$ as an induced proper subgraph. This means that H contains an induced subgraph F such that there is some vertex x in F for which $F \setminus x$ is a $2P_3$. But H is a linear forest, and so x has at most two neighbors in the $2P_3$, both being endpoints of the respective P_3's. Consequently, H contains an induced $2P_2 + P_1$ which means that there are infinitely many H-free 4-vertex-critical graphs by Lemma 10.

So, we may assume that H is $2P_3$-free and, in particular, $|H_1| \leq 6$ and $|H_2| \leq 2$.

Suppose that $|H_1| \geq 5$. Since there are infinitely many $2P_2 + P_1$-free obstructions, due to Lemma 11, we must have $k = 1$. This implies that H is a path on at most 6 vertices, and thus there are only finitely many H-free obstructions by Lemma 14.

We now assume that $|H_1| = 4$. If $|H_2| \geq 2$, H contains $2P_2 + P_1$ as an induced subgraph, and so Lemma 11 implies that there are infinitely many H-free obstructions. Thus, we may assume that $|H_2| \leq 1$ and so $H = P_4 + (k-1)P_1$. Consequently, the list of minimal H-free list obstructions is finite according to Lemma 19.

By Ramsey's theorem [24], and since a minimal obstruction is either K_4 or K_4-free, we may assume that $|H_1| \geq 2$. If $k \leq 2$ we are done, because H is a subgraph of P_6. So we may assume $k \geq 3$. If $|H_2| \geq 2$, then there are infinitely many obstructions, since H contains $2P_2 + P_1$ as a subgraph. Otherwise, H is a subgraph of $P_4 + (k-1)P_1$ and thus the list of minimal H-free list obstructions is finite, again by Lemma 19. □

We now come to the list-version of our main result. We prove Theorem 3, that we restate

Theorem 3. *Let H be a graph. There are only finitely many H-free minimal list-obstructions if and only if H is a subgraph of P_6 or $P_4 + kP_1$ for some $k \in \mathbb{N}$.*

Proof. Let H be a graph. If H contains a cycle or a claw, there is an infinite list of obstructions due to Lemma 11. Thus, we may assume that H is a linear forest. Let $H_1, H_2, \dots H_k$ be the components of H, ordered such that $|H_1| \geq |H_2| \geq \dots \geq H_k$.

If H contains an induced $2P_3$, there are infinitely many list-obstructions due to Lemma 11. In particular, $|H_1| \leq 6$ and $|H_2| \leq 2$. The rest of the proof can be done in complete analogy to the proof of Theorem 2. □

9 Proof of Theorem 6

In this section we prove the following result.

Theorem 6. *Let H be a connected graph. The 3-colorability problem admits a polynomial kernel of order $O(r^{O(1)})$ on the class of graphs that can be made H-free by the removal of r vertices if and only if H is a subgraph of P_6. The only if part holds under the assumption that NP $\not\subseteq$ coNP/poly.*

Our proof makes use of the following lower bound due to Dell and van Melkebeek [7] on the computational complexity of the r-SAT problem.[1] We use the version stated in the article by Jansen and Kratsch [17].

Theorem 20 (Dell and van Melkebeek [7])**.** *If NP $\not\subseteq$ coNP/poly then for any $\epsilon > 0$ and $q \geq 3$ there is no polynomial time algorithm which transforms an instance x of r-SAT on n variables into an equivalent instance x' of a decidable problem with bitsize $|x'| \in O(n^{r-\epsilon})$.*

We can now state the proof of Theorem 6.

Proof of Theorem 6. Throughout the proof we assume NP $\not\subseteq$ coNP/poly. In particular, P $\neq$ NP.

Let H be a connected graph. If H is a subgraph of P_6, there is a polynomial kernel by Theorem 3 and Theorem 5.

For the remainder of the proof we assume that H is not a subgraph of P_6. Let $\mathcal{G}$ be the class of H-free graphs. We need to prove that there is no polynomial kernel of order $O(r^{O(1)})$ for the 3-colorability problem on graphs in $\mathcal{G} + rv$.

First suppose that H is not a path. Then the 3-colorability problem is NP-hard on the class of H-free graphs [16, 18, 19, 21]. Hence, P $\neq$ NP implies that there is no polynomial time algorithm to solve the 3-colorability problem on $\mathcal{G}$. In particular, there is no polynomial kernel of order $O(r^{O(1)})$ for the 3-colorability problem on graphs in $\mathcal{G} + rv$.[2]

So, we may assume that H is a path. Since H is not a subgraph of P_6, H contains an induced P_7. In particular, H_r is H-free for all r, where H_r is the graph of the minimal list obstruction we defined in Section 3.2.

Let F be an r-SAT formula in CNF, say with n variables and m clauses. We construct a graph G which is 3-colorable if and only if F is satisfiable.

First we add a triangle on the three vertices x_1, x_2, x_3 to G. In every 3-coloring of G, if any, we may assume that x_i receives color i, $i = 1, 2, 3$. Thus, by adding an edge from a vertex v to x_i, say, we delete color option i from the list of v.

For each clause C_i we add one copy of H_{r+1} to G and call it H^i. We denote the j-th vertex of H^i by v^i_j. For each variable X_i of F we add two new vertices t_i and f_i and make them adjacent. Moreover, we add the edges $t_i x_3$ and $f_i x_3$ to G.

For $i = 1, \dots, m$ we add all possible edges between $\{v^i_1, v^i_{3r+2}\}$ and $\{x_2, x_3\}$. Moreover, for $i = 1, \dots, m$ and $j = 2, \dots, 3r + 2$ we add the edge $v^i_j x_2$ if $j \equiv 1 \mod 3$ and we add the edge $v^i_j x_3$ if $j \equiv 2 \mod 3$.

For $j = 1, \dots, r$ and $i = 1, \dots, m$ let L^i_j be the j-th literal in the clause C_i. Let k be such that L^i_j equals X_k or its negation. In the first case, we add the edge $t_k v^i_{3j}$ to G, and in the second case, we add the edge $f_k v^i_{3j}$. The construction of G is now complete, and it can obviously be implemented to run in polynomial time. Note that G is in the class $\mathcal{G} + (2n + 3)v$. We now claim that G is 3-colorable if and only if F is satisfiable.

To see this, first assume that G is 3-colorable, and let c be a 3-coloring of G. As mentioned above, we may assume that $c(x_i) = i$ for $i = 1, 2, 3$. In particular, c is a list coloring of the pair (G, L) where $L(x_i) = i$, $i = 1, 2, 3$, and $L(v) = \{1, 2, 3\} \setminus \{i : vx_i \in E(G)\}$ for all $v \notin \{x_1, x_2, x_3\}$.

From $L(t_i) = L(f_i) = \{1, 2\}$ and $t_i f_i \in E(G)$ it follows that $\{c(t_i), c(f_i)\} = \{1, 2\}$, $i = 1, \dots, n$. We define a truth assignment as follows: we set X_i to be TRUE if $c(t_i) = 2$ and FALSE if $c(f_i) = 2$, $i = 1, \dots, n$. It remains to show that this assignment satisfies F.

[1] In this problem, we are given a Boolean formula in conjunctive normal form and we need to decide whether there is a satisfying assignment. Each clause is the disjunction of exactly r literals.

[2] Such a kernel would reduce a graph in $\mathcal{G} = \mathcal{G} + 0v$ in polynomial time to an equivalent instance of constant size. This yields a polynomial time algorithm to solve the 3-colorability problem on $\mathcal{G}$, a contradiction.

To see this, suppose there is a clause C_i that is not satisfied. This means that L^i_j is false, $j = 1, \dots, r$. By construction, the vertex v^i_{3j} has a neighbor of color 1, namely t_k (or f_k) when L^i_j equals X_k (or the negation of X_k), $j = 1, \dots, r$. So, we may safely reduce the list $L(v^i_{3j})$ from $\{1, 2, 3\}$ to $\{2, 3\}$, $j = 1, \dots, r$. But now $(H^i, L|_{V(H^i)})$ is a list obstruction according to Section 3.2, in contradiction to the fact that c is a coloring of G.

Now we assume that there is a satisfying assignment of F. Due to symmetry, we may assume that all variables are set to TRUE: flipping the sign of the literals yields a graph that is isomorphic to G. We have to define a 3-coloring c of G.

We put $c(x_i) = i$ for $i = 1, 2, 3$. Moreover, we put $c(t_i) = 2$ and $c(f_i) = 1$ for $i = 1, \dots, n$. Like above, it helps to think of c as a list coloring of the pair (G, L) where $L(x_i) = i$, $i = 1, 2, 3$, and $L(v) = \{1, 2, 3\} \setminus \{i : vx_i \in E(G)\}$ for all $v \notin \{x_1, x_2, x_3\}$.

After updating once from the vertex set $\{t_k, f_k : k = 1, \dots, n\}$ we obtain a new list system L'. The system L' equals L except for the following changes: for each edge of the form $v^i_{3j}t_k$ we have $L'(v^i_{3j}) = \{1, 3\}$ and for each edge of the form $v^i_{3j}f_k$ we have $L'(v^i_{3j}) = \{2, 3\}$.

It remains to prove that for each H^i we can extend the partial coloring c to H^i, $i = 1, \dots, m$. To this end, let $J \subseteq \{1, \dots, r\}$ be such that $L'(v^i_{3j}) = \{1, 3\}$. Since C_i is satisfied by our assignment, $J \neq \emptyset$. We put $c(v^i_{3j-1}) = 2$, $c(v^i_{3j}) = 1$, and $c(v^i_{3j+1}) = 3$ for all $j \in J$. Now consider the graph $\hat{H}^i := H^i \setminus \{v^i_{3j-1}, v^i_{3j}, v^i_{3j+1} : j \in J\}$. The pair $(\hat{H}^i, L'|_{\hat{H}^i})$ has a list coloring since it is obtained from the minimal list obstruction described in Section 3.2 by the removal of at least three vertices. Extend c to $\hat{H}^i$ according to such a list coloring, and observe that there are no conflicts with the colors of the removed set $\{v^i_{3j-1}, v^i_{3j}, v^i_{3j+1} : j \in J\}$ by construction. Summing up, c is a list coloring of (G, L') and thus a 3-coloring of G.

It remains to draw the conclusion that there is no polynomial kernel of order $O(r^{O(1)})$ for the 3-colorability problem on graphs in the class $\mathcal{G} + rv$. Suppose there was a polynomial kernel. Then we could compress the r-SAT problem as follows: given a formula F with n variables, construct the graph G in the class $\mathcal{G} + (2n+3)v$. Now compute a kernel, which is then of order $O(n^{O(1)})$ and, obviously, of bitsize $O(n^{O(1)})$. But this contradicts Theorem 20 and thus completes the proof. □

10 Further research

In this section we describe two open problems that are natural next steps to the results presented here.

As mentioned in the introduction, in the proof of Theorem 1 we explicitly give the complete list of 4-vertex-critical P_6-free graphs. Maffray and Morel [22] gave a similar characterization in the P_5-free case and derived a structural characterization of the 3-colorable P_5-free graphs. From this structural characterization they were able to derive linear time algorithms to both (a) recognize 3-colorable P_5-free graphs and (b) compute a maximum weight stable set in the class of 3-colorable P_5-free graphs. The question is whether the same can be achieved in the P_6-free case. Certainly a solution to this problem will be more involved than in the P_5-free case, but the hope is that some tricks from the paper of Maffray and Morel [22] might carry over.

Another possible direction for further research is to characterize all graphs H such that there is a polynomial kernel for 3-colorability parameterized by the number of vertices that have to be deleted to make the input graph H-free. We solved the connected case in Theorem 6, so the case of disconnected graphs H is left open.

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