# Obstructions for three-coloring graphs without induced paths on six vertices

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**Abstract**

We prove that there are 24 4-critical $P_6$-free graphs, and give the complete list. We remark that, if $H$ is connected and not a subgraph of $P_6$, there are infinitely many 4-critical $H$-free graphs. Our result answers questions of Golovach et al. and Seymour.

## 1 Introduction

A *k-coloring* of a graph $G = (V, E)$ is a mapping $c : V \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for all edges $uv \in E$. If a $k$-coloring exists, we say that $G$ is *k-colorable*. We say that $G$ is *k-chromatic* if it is $k$-colorable but not $(k-1)$-colorable. A graph is called *k-critical* if it is $k$-chromatic, but every proper subgraph is $(k-1)$-colorable. For example, the class of 3-critical graphs is the family of all chordless odd cycles. The characterization of critical graphs is a notorious problem in the theory of graph coloring, and also the topic of this paper.

Since it is NP-hard to decide whether a given graph admits a $k$-coloring, assuming $k \geq 3$, there is little hope of giving a characterization of the $(k+1)$-critical graphs that is useful for algorithmic purposes. The picture changes if one restricts the structure of the graphs under consideration.

Let a graph $H$ and a number $k$ be given. An *H-free* graph is a graph that does not contain $H$ as an induced subgraph. We say that a graph $G$ is *k-critical H-free* if $G$ is $H$-free, $k$-chromatic, and every $H$-free proper subgraph of $G$ is $(k-1)$-colorable. In this paper we stick to the case of 4-critical graphs; these graphs we may informally call *obstructions*.

Bruce et al. [2] proved that there are exactly six 4-critical $P_5$-free graphs, where $P_t$ denotes the path on $t$ vertices. Randerath et al. [16] have shown that the only 4-critical $P_6$-free graph without a triangle is the Grötzsch graph (i.e., the graph $F_{18}$ in Fig. 2). More recently, Hell and Huang [10] proved that there are four 4-critical $P_6$-free graphs without induced four-cycles.

In view of these results, Golovach et al. [9] posed the question of whether the list of 4-criticial $P_6$-free graphs is finite (cf. *Open Problem 4* in [9]). In fact, they ask whether there is a certifying algorithm for the 3-colorability problem in the class of $P_6$-free graphs, which is an immediate consequence of the finiteness of the list. Our main result answers this question affirmatively.

**1.1.** *There are exactly 24 4-critical $P_6$-free graphs.*

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These 24 graphs, which we denote here by $F_1$-$F_{24}$, are shown in Fig. 1 and 2. The list contains several familiar graphs, e.g., $F_1$ is $K_4$, $F_2$ is the 5-wheel, $F_3$ is the Moser-spindle, and $F_{18}$ is the Grötzsch graph. The adjacency lists of these graphs can be found in the Appendix.

We also determined that there are exactly 80 4-vertex-critical $P_6$-free graphs (details on how we obtained these graphs can be found in the Appendix). Table 1 gives an overview of the counts of all 4-critical and 4-vertex-critical $P_6$-free graphs. All of these graphs can also be obtained from the *House of Graphs* [1] by searching for the keywords "4-critical P6-free" or "4-vertex-critical P6-free" where several of their invariants can be found.

In Section 8 we show that there are infinitely many 4-critical $P_7$-free graphs using a construction due to Pokrovskiy [15]. Note that there are infinitely many 4-critical claw-free graphs. For example, this follows from the existence of 4-regular bipartite graphs of arbitrary large girth (cf. [12] for an explicit construction of these), whose line graphs are then 4-chromatic. Also, there are 4-chromatic graphs of arbitrary large girth, which follows from a classical result of Erdős [5]. This together with 1.1 yields the following dichotomy theorem, which answers a question of Seymour [17].

**1.2.** *Let $H$ be a connected graph. There are finitely many 4-critical $H$-free graphs if and only if $H$ is a subgraph of $P_6$.*

We will next give a sketch of the proof of our main result, thereby explaining the structure of this paper.

| Vertices | Critical graphs | Vertex-critical graphs |
|---|---|---|
| 4 | 1 | 1 |
| 6 | 1 | 1 |
| 7 | 2 | 7 |
| 8 | 3 | 6 |
| 9 | 4 | 16 |
| 10 | 6 | 34 |
| 11 | 2 | 3 |
| 12 | 1 | 1 |
| 13 | 3 | 9 |
| 16 | 1 | 2 |
| total | 24 | 80 |

Table 1: Counts of all 4-critical and 4-vertex-critical $P_6$-free graphs.

## 1.1 Sketch of the proof

Given a 4-critical $P_6$-free graph, our aim is to show that it is contained in our list of 24 graphs. Our proof is based on the contraction (and uncontraction) of a particular kind of subgraph called *tripod*. Tripods have been used before in the design of 3-coloring algorithms for $P_7$-free graphs [3]. In Section 2 tripods are defined, and it is shown that contracting a maximal tripod to a single triangle is a *safe* operation for our purpose.

When all maximal tripods are just single triangles, we are left with a $(P_6, \text{diamond})$-free graph, a *diamond* being the graph obtained by removing an edge from $K_4$. The second step of our proof consists of determining all 4-critical $(P_6, \text{diamond})$-free graphs, which we do in Section 3. Our proof is computer-aided, and builds on a substantial strengthening of a method by Hoàng et al. [11].

In Section 4 we show the following. Let $G$ be a non-3-colorable $P_6$-free graph that is obtained from another graph $G'$ by contracting a tripod. If $G$ contains one of our 24 obstructions, then so does $G'$ (we need a few additional assumptions if $G$ contains $K_4$, but we will not list them here). The proof is done by a structural analysis by hand, and it does not use a computer.

Finally, in Section 5 we deal with the exceptionally difficult case of uncontracting a triangle in $K_4$. For this, we again use an automatic proof, though completely different than in the $(P_6, \text{diamond})$-free case. We design an algorithm that performs an exhaustive generation of all possible 1-vertex extensions of tripods

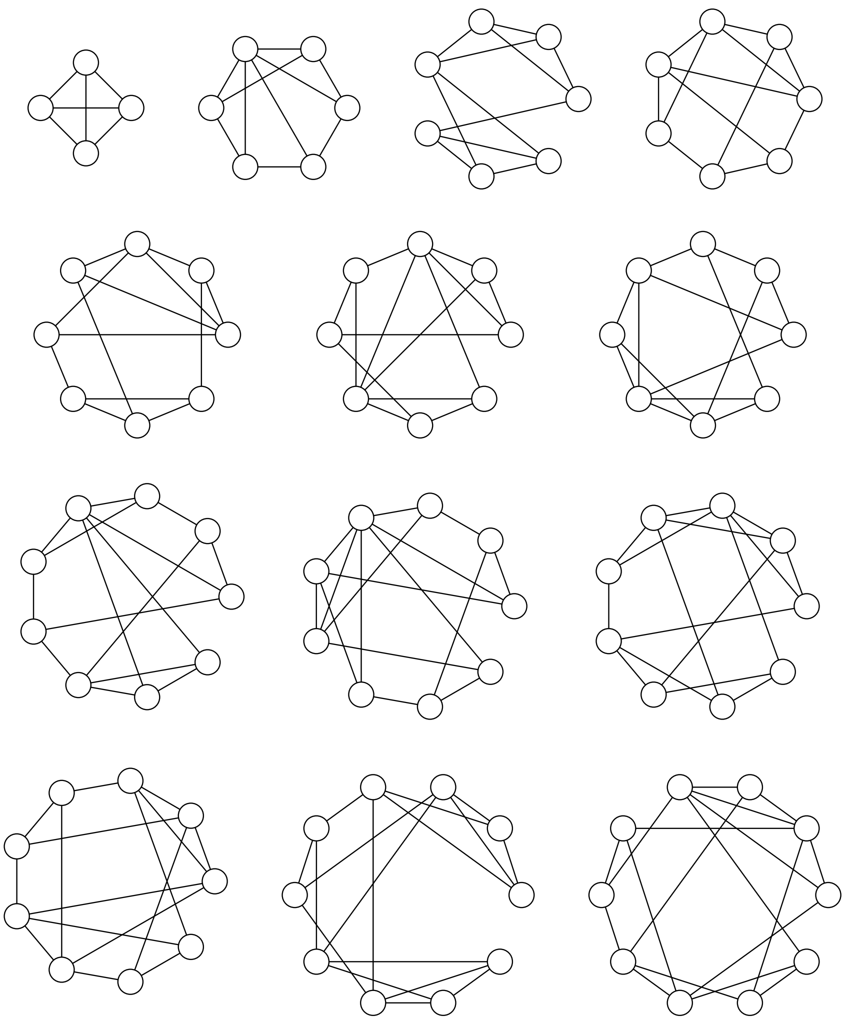

Figure 1: The graphs $F_1$ to $F_{13}$, in reading direction

that are 4-critical $P_6$-free. The algorithm mimicks the way that a tripod can be traversed, thereby applying a set of strong pruning rules that exploit the minimality of the obstruction.

We wrap up the whole proof in Section 7.

As mentioned earlier, in Section 8 we show that there are infinitely many 4-critical $P_7$-free graphs, which results in our dichotomy theorem.

# 2 Tripods

A *tripod* in a graph $G$ is a triple $T = (A_1, A_2, A_3)$ of disjoint stable sets with the following properties:

(a) $A_1 \cup A_2 \cup A_3 = \{v_1, \ldots, v_k\}$;

(b) $v_i \in A_i$ for $i = 1, 2, 3$;

(c) $v_1v_2v_3$ is a triangle, the *root* of $T$; and

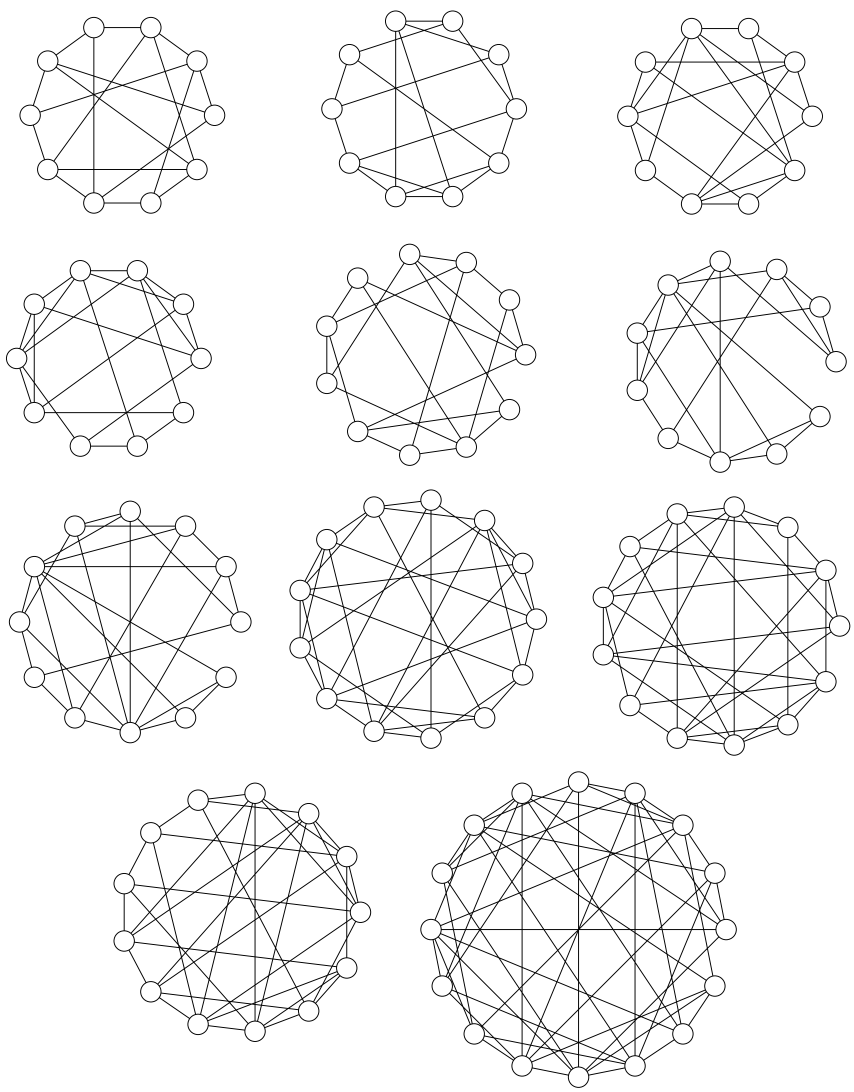

Figure 2: The graphs $F_{14}$ to $F_{24}$, in reading direction

(d) for all $i \in \{1,2,3\}$, $\{\ell, k\} = \{1,2,3\} \setminus \{i\}$, and $j \in \{4,\dots,k\}$ with $v_j \in A_i$, the vertex $v_j$ has neighbors in both $\{v_1,\dots,v_{j-1}\} \cap A_\ell$ and $\{v_1,\dots,v_{j-1}\} \cap A_k$.

Assuming that $G$ admits a 3-coloring, it follows right from the definition above that each $A_i$ is contained in a single color class. Moreover, since $v_1v_2v_3$ is a triangle, $A_1, A_2, A_3$ are pairwise contained in distinct color classes.

To better reference the ordering of the tripod, we put $t(v_1) = t(v_2) = t(v_3) = 0$, and $t(v_i) = i - 3$ for all $4 \le i \le k$. For each $u \in A_i$, let $n_j(u)$ be the neighbor $v$ of $u$ in $A_j$ with $t(v)$ minimum, where $i, j \in \{1,2,3\}, i \neq j$. We write $T(t) = G|\{v \in V(T) : t(v) \le t\}$, i.e., the subgraph induced by $G$ on the vertex set $\{v \in V(T) : t(v) \le t\}$. Moreover, we write $T_i$ for the graph $G|(A_j \cup A_k)$ where $\{i,j,k\} = \{1,2,3\}$, and finally $T_i(t)$ for the graph $G|\{v \in A_j \cup A_k : t(v) \le t\}$.

We call a tripod $(A_1, A_2, A_3)$ *maximal* in a given graph if no further vertex can be added to any set $A_i$ without violating the tripod property.

## 2.1 Contracting a tripod

By *contracting* a tripod $(A_1, A_2, A_3)$ we mean the operation of identifying each $A_i$ to a single vertex $a_i$, for all $i = 1, 2, 3$. We then make $a_i$ adjacent to the union of neighbors of the vertices in $A_i$, for all $i = 1, 2, 3$.

The neighborhood of a vertex $v$ in a graph $G$ we denote $N_G(v)$. If $G$ is clear from the context we might also omit $G$ in the subscript.

**2.1.** *Let $G$ be a graph with a maximal tripod $T$ such that no vertex of $G$ has neighbors in all three classes of $T$. Let $G'$ be the graph obtained from $G$ by contracting $T$. Then the following holds.*

*(a) The graph $G$ is 3-colorable if and only if $G'$ is 3-colorable, and*

*(b) if $G$ is $P_6$-free, $G'$ is $P_6$-free.*

*Proof.* Assertion (a) follows readily from the definition of a tripod, so we just prove (b). For this, suppose that $G$ is $P_6$-free but $G'$ contains an induced $P_6$, say $P = v_1\text{-}\ldots\text{-}v_6$. Let $T = (A_1, A_2, A_3)$, and let $a_i$ be the vertex of $G'$ the set $A_i$ is contracted to, for $i = 1, 2, 3$.

Since $P$ is an induced path, it cannot contain all three of $a_1, a_2, a_3$. Moreover, if $P$ contains neither of $a_1, a_2, a_3$, then $G$ contains a $P_6$, a contradiction.

Suppose that $P$ contains, say, $a_1$ and $a_2$. We may assume that $a_1 = v_i$ and $a_2 = v_{i+1}$ for some $1 \leq i \leq 3$. If $i = 1$, pick $b \in A_1$ and $c \in A_2 \cap N_G(v_3)$ with minimum distance in $T_3$. Otherwise, if $i \geq 2$, pick $b \in A_1 \cap N_G(v_{i-1})$ and $c \in A_2 \cap N_G(v_{i+2})$ again with minimum distance in $T_3$. In both cases, let $Q$ be the shortest path between $b$ and $c$ in $T_3$. Due to the choice of $b$ and $c$, the induced path $v_1\text{-}\ldots\text{-}b\text{-}Q\text{-}c\text{-}\ldots\text{-}v_6$ is induced in $G$, which means $G$ contains a $P_6$, a contradiction.

So, we may assume that $P$ contains only one of $a_1, a_2, a_3$, say $v_i = a_1$ for some $1 \leq i \leq 3$. We obtain an immediate contradiction if $i = 1$, so suppose that $i \geq 2$. Since $v_{i+2}$ is not contained in $T$, we may assume that $v_{i+2}$ is anticomplete to $A_2$ in $G$. Pick $b \in A_1 \cap N_G(v_{i-1})$ and $c \in A_1 \cap N_G(v_{i+1})$ such that the distance in $T_3$ between $b$ and $c$ is minimum. Let $Q$ be a shortest path in $T_3$ between $b$ and $c$. Since $v_i v_{i+2} \notin E(G')$, $v_{i+2}$ is anticomplete to $A_1$ and thus to $V(Q)$ in $G$. If $b = c$, then $v_1\text{-}\ldots\text{-}v_{i-1}\text{-}b\text{-}v_{i+1}\text{-}\ldots\text{-}v_6$ is a $P_6$ in $G$, a contradiction. Otherwise, the induced path $v_{i-1}\text{-}b\text{-}Q\text{-}c\text{-}v_{i+1}\text{-}v_{i+2}$ is induced in $G$ and contains at least six vertices, which is also contradictory. □

# 3 Diamond-free obstructions

Recall that a diamond is the graph obtained by removing an edge from $K_4$. After successively contracting all maximal tripods in a graph, we are left with a diamond-free graph. In this section we prove the following statement.

**3.1.** *There are exactly six 4-critical $(P_6, \text{diamond})$-free graphs.*

These graphs are $F_1$, $F_{11}$, $F_{14}$, $F_{16}$, $F_{18}$, and $F_{24}$ in Fig. 1 and 2.

The proof of 3.1 is computer-aided, and builds upon a method recently proposed by Hoàng et al. [11]. With this method they have shown that there is a finite number of 5-critical $(P_5, C_5)$-free graphs. The idea is to automatize the large number of necessary case distinctions, resulting in an exhaustive enumeration algorithm. Since we have to deal with a graph class which is substantially less structured, we need to significantly extend their method.

## 3.1 Preparation

In order to prove 3.1, we make use of the following tools.

Let $G$ be a $k$-colorable graph. We define the *$k$-hull* of $G$, denoted $G_k$, to be the graph with vertex set $V(G)$ where two vertices $u, v$ are adjacent if and only if there is no $k$-coloring of $G$ where $u$ and $v$ receive the same color. Note that $G_k$ is a simple supergraph of $G$, since adjacent vertices can never receive the same color in any coloring. Moreover, $G_k$ is $k$-colorable.

It is easy to see that a $k$-critical graph cannot contain two distinct vertices, $u$ and $v$ say, such that $N(u) \subseteq N(v)$. The following observation is a proper generalization of this fact.

**3.2.** *Let $G = (V, E)$ be a $k$-vertex-critical graph and let $U, W$ be two non-empty disjoint vertex subsets of $G$. Let $H := (G - U)_{k-1}$. If there exists a homomorphism $\phi : G|U \mapsto H|W$, then $N_G(u) \setminus U \not\subseteq N_H(\phi(u))$ for some $u \in U$.*

Note that, in the statement of 3.2, $H$ is well-defined since $G$ is $k$-vertex-critical.

*Proof of 3.2.* Suppose that $N_G(u) \setminus U \subseteq N_H(\phi(u))$ for all $u \in U$. Fix some $(k-1)$-coloring $c$ of $H$. In particular, for each $u \in U$, the color of $\phi(u)$ is different from that of any member of $N_H(\phi(u))$.

We now extend $c$ to a $(k-1)$-coloring of $G$ by giving any $u \in U$ the color $c(\phi(u))$. It suffices to show that this is a proper coloring. Clearly there are no conflicts between any two vertices of $U$, since $\phi$ is a homomorphism. Let $u \in U$ and $v \in N_G(u) \setminus U$ be arbitrary. Since $N_G(u) \setminus U \subseteq N_H(\phi(u))$, $c(v) \neq c(\phi(u))$, and so $u$ and $v$ receive distinct colors. But this contradicts with the assumption that $G$ is a $k$-vertex-critical graph. □

We make use of 3.2 in the following way. Assume that $G$ is a $(k-1)$-colorable graph that is an induced subgraph of some $k$-vertex-critical graph $G'$. Pick two non-empty disjoint vertex subsets $U, W \subseteq V$ of $G$, and let $H := (G-U)_{k-1}$. Assume there exists a homomorphism $\phi : G|U \mapsto H|W$ such that $N_G(u) \setminus U \subseteq N_H(\phi(u))$ for all $u \in U$. Then there must be some vertex $x \in V(G') \setminus V(G)$ which is adjacent to some $u \in U$ but non-adjacent to $\phi(u)$ in $G'$. Moreover, $x$ is non-adjacent to $\phi(u)$ in the graph $(G' - U)_{k-1}$.

We also make use of the following well-known fact.

**3.3.** *A $k$-vertex-critical graph has minimum degree at least $k$.*

Another fact we need is the following.

**3.4.** *Any $(P_6, \text{diamond})$-free 4-critical graph other than $K_4$ contains an induced $C_5$.*

*Proof.* By the Strong Perfect Graph Theorem [4], every 4-critical graph different from $K_4$ must contain an odd hole or an anti-hole as an induced subgraph. A straightforward argumentation shows that only the 5-hole, $C_5$, can possibly appear. □

## 3.2 The enumeration algorithm

Generally speaking, our algorithm constructs a graph $G'$ with $n+1$ vertices from a graph $G$ with $n$ vertices by adding a new vertex and connecting it to vertices of $G$ in all possible ways. So, all graphs constructed from $G$ contain $G$ as an induced subgraph. Since 3-colorability and $(P_6, \text{diamond})$-freeness are both hereditary properties, we do not need to expand $G$ if it is not 3-colorable, contains a $P_6$ or a diamond.

We use Algorithm 1 below to enumerate all $(P_6, \text{diamond})$-free 4-critical graphs. In order to keep things short, we use the following conventions for a graph $G$. We call a pair $(u, v)$ of distinct vertices for which $N_G(u) \subseteq N_{(G-u)_3}(v)$ *similar vertices.* Similarly, we call a 4-tuple $(u, v, u', v')$ of distinct vertices with $uv, u'v' \in E(G)$ such that $N_G(u) \setminus \{v\} \subseteq N_{(G-\{u,v\})_3}(u')$ and $N_G(v) \setminus \{u\} \subseteq N_{(G-\{u,v\})_3}(v')$ *similar edges.* Finally, we define *similar triangles* in an analogous fashion.

**Algorithm 1** Generate $(P_6, \text{diamond})$-free 4-critical graphs

```
1: Let 𝓕 be an empty list
2: Add K_4 to the list 𝓕
3: Construct(C_5)  // i.e. perform Algorithm 2
4: Output 𝓕
```

We now prove that Algorithm 1 is correct.

**Algorithm 2** Construct(Graph $G$)

1: **if** $G$ is $(P_6, \text{diamond})$-free AND not generated before **then**
2: **if** $G$ is not 3-colourable **then**
3: **if** $G$ is 4-critical $P_6$-free **then**
4: add $G$ to the list $\mathcal{F}$
5: **end if**
6: return
7: **else**
8: **if** $G$ contains similar vertices $(u, v)$ **then**
9: **for** every graph $H$ obtained from $G$ by attaching a new vertex $x$ and incident edges in all possible ways, such that $ux \in E(H)$, but $vx \notin E((H - u)_3)$ **do**
10: Construct($H$)
11: **end for**
12: **else if** $G$ contains a vertex $u$ of degree at most 2 **then**
13: **for** every graph $H$ obtained from $G$ by attaching a new vertex $x$ and incident edges in all possible ways, such that $ux \in E(H)$ **do**
14: Construct($H$)
15: **end for**
16: **else if** $G$ contains similar edges $(u, v, u', v')$ **then**
17: **for** every graph $H$ obtained from $G$ by attaching a new vertex $x$ and incident edges in all possible ways, such that $ux \in E(H)$ and $u'x \notin E((H - \{u, v\})_3)$, or $vx \in E(H)$ and $v'x \notin E((H - \{u, v\})_3)$ **do**
18: Construct($H$)
19: **end for**
20: **else if** $G$ contains similar triangles $(u, v, w, u', v', w')$ **then**
21: **for** every graph $H$ obtained from $G$ by attaching a new vertex $x$ and incident edges in all possible ways, such that $ux \in E(H)$ and $vx \notin E((H - \{u, v, w\})_3)$, $vx \in E(H)$ and $v'x \notin E((H - \{u, v, w\})_3)$, or $wx \in E(H)$ and $w'x \notin E((H - \{u, v, w\})_3)$ **do**
22: Construct($H$)
23: **end for**
24: **else**
25: **for** every graph $H$ obtained from $G$ by attaching a new vertex $x$ and incident edges in all possible ways **do**
26: Construct($H$)
27: **end for**
28: **end if**
29: **end if**
30: **end if**

**3.5.** *Assume that Algorithm 1 terminates, and outputs the list of graphs $\mathcal{F}$. Then $\mathcal{F}$ is the list of all $(P_6, \text{diamond})$-free 4-critical graphs.*

*Proof.* In view of lines 1 and 3 of Algorithm 2, it is clear that all graphs of $\mathcal{F}$ are 4-critical $(P_6, \text{diamond})$-free. So, it remains to prove that $\mathcal{F}$ contains all $(P_6, \text{diamond})$-free 4-critical graphs. To see this, we first prove the following claim.

**3.6.** *For every $(P_6, \text{diamond})$-free 4-critical graph $F$ other than $K_4$, Algorithm 2 applied to $C_5$ generates an induced subgraph of $F$ with $i$ vertices for every $5 \leq i \leq |V(F)|$.*

We prove this inductively, as an invariant of our algorithm. Due to 3.4, we know that $F$ contains an induced $C_5$, so the claim holds true for $i = 5$.

So assume that the claim is true for some $i \geq 5$ with $i < |V(F)|$. Let $G$ be the induced subgraph of $F$ with $|V(G)| = i$. First assume that $G$ contains similar vertices $(u, v)$. Then, by 3.2, $N_F(u) \setminus U \not\subseteq N_{(F-u)_3}(v)$. Hence, there is some vertex $x \in V(F) \setminus V(G)$ which is adjacent to $u$ in $F$, but not to $v$ in $(F-u)_{k-1}$. Following the statement of line 10, Construct($F|(V(G) \cup \{x\})$) is called. We omit the discussion of the lines 16 and 20, as they are analogous.

So assume that $G$ contains a vertex $u$ of degree at most 2. Then, since the minimum degree of any 4-vertex-critical graph is at least 3, there is some vertex $x \in V(F) \setminus V(G)$ adjacent to $u$. Following the statement of line 26, Construct($F|(V(G) \cup \{x\})$) is called.

Finally, if none of the above criteria apply to $G$, the algorithm attaches a new vertex to $G$ in all possible ways, and calls Construct for all of these new graphs. Since $|V(F)| > |V(G)|$, among these graphs there is some induced subgraph of $F$, and of course this graph has $i + 1$ vertices. This completes the proof of 3.6.

Given that the algorithm terminates and $K_4$ is added to the list $\mathcal{F}$, 3.6 implies that $\mathcal{F}$ must contain all 4-critical ($P_6$, diamond)-free graphs. □

We implemented this algorithm in C with some further optimizations. To make sure that no isomorphic graphs are accepted (cf. line 1 of Algorithm 2), we use the program `nauty` [13, 14] to compute a canonical form of the graphs. We maintain a list of the canonical forms of all non-isomorphic graphs which were generated so far and only accept a graph if it was not generated before (and then add its canonical form to the list).

Our program does indeed terminate (in about 2 seconds), and outputs the six graphs $F_1$, $F_{11}$, $F_{14}$, $F_{16}$, $F_{18}$, and $F_{24}$ from Fig. 1 and 2. Together with 3.5 this proves 3.1. Let us stress the fact that in order for the algorithm to terminate, all proposed expansion rules are needed.

Table 2 shows the number of non-isomorphic graphs generated by the program. The source code of the program can be downloaded from [6] and in the Appendix we describe how we extensively tested the correctness of our implementation.

The second and third author also extended this algorithm which allowed to determine all $k$-critical graphs for several other cases as well (see [8]).

| $\|V(G)\|$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # graphs generated | 1 | 4 | 16 | 55 | 130 | 230 | 345 | 392 | 395 | 279 | 211 | 170 |
| $\|V(G)\|$ | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| # graphs generated | 112 | 95 | 74 | 53 | 40 | 32 | 20 | 15 | 12 | 3 | 1 | 0 |

Table 2: Counts of the number of non-isomorphic ($P_6$, diamond)-free graphs generated by our implementation of Algorithm 1.

# 4 Uncontracting a triangle to a tripod

Let $G$ be a $P_6$-free graph. Let $C$ be a hole in $G$. A *leaf* for $C$ is a vertex $v \in V(G) \setminus V(C)$ with exactly one neighbor in $V(G)$. Similarly, a *hat* for $C$ is a vertex in $V(G) \setminus V(C)$ with exactly two neighbors $u, v \in V(C)$, where $u$ is adjacent to $v$.

The following observation is immediate from the fact that $G$ is $P_6$-free.

**4.1.** *No $C_6$ in $G$ has a leaf or a hat.*

Let $T = (A_1, A_2, A_3)$ be a maximal tripod of $G$ with $A_1 \cup A_2 \cup A_3 = \{v_1, \ldots, v_k\}$.

**4.2.** *The graph $T_i(t)$ is connected, for all $i \in \{1, 2, 3\}$ and $0 \leq t \leq k$.*

*Proof.* This follows readily from the definition of a tripod. □

**4.3.** *Let $a \in A_1$, and let $y, z \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$ such that $a$-$y$-$z$ is an induced path, and $z$ is anticomplete to $A_2 \cup A_3$. Then $(A_2 \cup A_3) \setminus N(a)$ is stable, and in particular, for $i = 2, 3$ there exist $n_i \in N(a) \cap A_i$ such that $n_2$ is adjacent to $n_3$.*

*Proof.* By the maximality of the tripod, $y$ is anticomplete to $A_2 \cup A_3$. Suppose there are $p_i \in A_i \setminus N(a)$, $i = 2, 3$, such that $p_2$ is adjacent to $p_3$. Since $T_1$ is connected, we can choose $p_2, p_3$ such that, possibly exchanging $A_2$ and $A_3$, $p_2$ has a neighbor $q_3$ in $A_3 \cap N(a)$. But now $z$-$y$-$a$-$q_3$-$p_2$-$p_3$ is a $P_6$, a contradiction. Since $T_3$ is connected, the second statement of the theorem follows. □

A *2-edge matching* are two disjoint edges $ab, cd$ where $ac, bd$ are non-edges.

**4.4.** *Let $X$ be a stable set in $V(G) \setminus (A_1 \cup A_2 \cup A_3)$, such that for every $x, x' \in X$ there exists $p \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$ such that $p$ is anticomplete to $A_1$ and adjacent to exactly one of $x, x'$. Assume that there is a 2-edge matching $ax, a'x'$ between $A_1$ and $X$. Then*

*(a) there do not exists $n_2 \in A_2$, and $n_3 \in A_3$ such that $\{a, a'\}$ is complete to $\{n_2, n_3\}$, and*

*(b) there exists $a'' \in A_1$, with $t(a'') < \max(t(a), t(a'))$ such that $a''$ is complete to $X \cap (N(a) \cup N(a'))$.*

*Proof.* Suppose $ax, a'x'$ is such a matching. We may assume that $xp$ is an edge. Let $P$ be an induced path from $a$ to $a'$ with interior in $A_2 \cup A_3$. Such a path exists since $T_1$ is connected, and both $a, a'$ have neighbors in $A_2 \cup A_3$. If $P$ has at least three edges, then $x$-$a$-$P$-$a'$-$x'$ is a $P_6$, so we may assume that $a, a'$ have a common neighbor $n_2 \in A_2$. If $p$ is non-adjacent to $n_2$, then $p$-$x$-$a$-$n_2$-$a'$-$x'$ is a $P_6$, a contradiction. So $p$ is adjacent to $n_2$, and therefore $p$ has no neighbor in $A_3$. By symmetry, $a, a'$ have no common neighbor in $A_3$, and so (a) follows.

Since $a, a'$ do not have a common neighbor in $A_3$, there is an induced path $a$-$b$-$c$-$d$-$a'$ from $a$ to $a'$ in $T_2$. Since $z$-$a$-$b$-$c$-$d$-$a'$ and $a$-$b$-$c$-$d$-$a'$-$z'$ are not a $P_6$ for any $z \in N(a) \setminus N(a')$, and $z' \in N(a') \setminus N(a)$, we deduce that $c$ is complete to $(N(a) \setminus N(a')) \cup (N(a') \setminus N(a))$. We may assume that there exists $x'' \in X \cap N(a) \cap N(a')$ such that $c$ is non-adjacent to $x''$, for otherwise (b) holds. Now if $p$ is non-adjacent to $x''$, then $p$-$x$-$c$-$d$-$a'$-$x''$ is a $P_6$, and if $p$ is adjacent to $x''$, then $p$-$x''$-$a$-$b$-$c$-$x'$ is a $P_6$, in both cases a contradiction. This proves (b). □

**4.5.** *Let $X, Y$ be two disjoint stable sets in $V(G) \setminus (A_1 \cup A_2 \cup A_3)$, such that every vertex of $X \cup Y$ has a neighbor in $A_1$. Moreover, assume that the following assertions hold.*

1. *For every $x \in X$ and $y \in Y$, either*
   - *(i) $x$ is adjacent to $y$,*
   - *(ii) $x$ has a neighbor in $V(G)$ anticomplete to $A_1$, or*
   - *(iii) $y$ has a neighbor in $V(G)$ anticomplete to $A_1$.*
2. *For every $x, x' \in X$ there exists $p \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$ such that*
   - *(i) $p$ is anticomplete to $A_1$, and*
   - *(ii) $p$ is adjacent to exactly one of $x, x'$.*
3. *The above assertion holds for $Y$ in an analogous way.*
4. *Let $u, v \in X \cup Y$ be distinct and non-adjacent. Then $N(u) \setminus A_1$ and $N(v) \setminus A_1$ are incomparable.*

*Then either*

*(a) there is a vertex $p \in A_1$ which is complete to $X \cup Y$, or*

*(b) there exist $c, d \in A_1$, $p \in A_2$ and $q \in A_3$, such that $p$ and $q$ are adjacent, $c$ is complete to $X$, $d$ is complete to $Y$, and $\{c, d\}$ is complete to $\{p, q\}$.*

*Proof.* After deleting all vertices of $V(G) \setminus (X \cup Y \cup A_1 \cup A_2 \cup A_3)$ with a neighbor in $A_1$ (this does not change the hypotheses or the outcomes), we may assume that no vertex of $V(G) \setminus (A_2 \cup A_3 \cup X \cup Y)$ has a neighbor in $A_1$.

*There exist $a, b \in A_1$ such that $a$ is complete to $X$, and $b$ is complete to $Y$.* (1)

To see (1), it is enough to show that $a$ exists, by symmetry. So, suppose not that such an $a$ does not exist. Pick $a \in A_1$ with $N(a) \cap X$ maximal, and note that $a$ is not complete to $X$ by assumption. By assumption, there exists $a' \in A_1$ and $x, x' \in X$ such that $ax, a'x'$ is a 2-edge matching. But now by 4.4.(b), there exists $a'' \in A_1$ complete to $(N(a) \cap X) \cup x'$, contrary to the choice of $a$. This proves (1).

We may assume that no vertex of $A_1$ is complete to $X \cup Y$, for otherwise 4.5.(a) holds. Moreover, we may assume that there exist $x \in X$, and $y \in Y$ such that $ax, by$ is a 2-edge matching. We choose $a, b$ with $t(a) + t(b)$ minimum, and subject to that $x$ and $y$ are chosen adjacent if possible.

*There is no $p \in A_1$, with $t(p) < \max(t(a), t(b))$ such that $p$ is complete to $(X \setminus N(b)) \cup (Y \setminus N(a))$.* (2)

Suppose such a $p$ exists. We may assume that $t(a) > t(b)$, and hence $t(p) < t(a)$. By the choice of $a$ and $b$, $p$ is not complete to $X$, and so there is a 2-edge matching between $\{b, p\}$ and $X$. Thus by 4.4.(b), there exists a vertex $p'$ with $t(p') < \max(t(b), t(p)) < t(a)$ that is complete to $X$, again contrary to the choice of $a$ and $b$. This proves (2).

*Either $a$ is adjacent to $n_2(b)$, or $b$ is adjacent to $n_2(a)$.* (3)

Suppose that this is false. We may assume that $t(n_2(a)) > t(n_2(b))$. Let $P$ be an induced path from $n_2(a)$ to $n_2(b)$ in $T_3(t(n_2(a)))$. Then $n_2(a)$ is the unique neighbor of $a$ in $P$. Since $a$-$n_2(a)$-$P$-$n_2(b)$ is not a $P_6$, we may deduce that $P$ has length two, say $P = n_2(a)$-$p$-$n_2(b)$. Moreover, since $x'$-$a$-$n_2(a)$-$p$-$n_2(b)$-$b$ is not a $P_6$ for any $x' \in X \setminus N(b)$, we know that $X \setminus N(b)$ is complete to $p$. Finally, since $y'$-$b$-$n_2(b)$-$p$-$n_2(a)$-$a$ is not a $P_6$ for any $y' \in Y \setminus N(a)$, $p$ is complete to $Y \setminus N(a)$. But since $p \in T_3(t(n_2(a))$, we know that $t(p) < t(a) \leq \max(t(a), t(b))$, contrary to (2). This proves (3).

By (3) and using the symmetry between $A_2$ and $A_3$, we may deduce that for $i = 2, 3$ there exists $n_i \in A_i$ such that $\{a, b\}$ is complete to $\{n_2, n_3\}$, and each $n_i$ is the smallest neighbor of one of $a, b$ in $A_i$ w.r.t. their value of $t$. We may assume that $n_2$ is non-adjacent no $n_3$, for otherwise 4.5.(b) holds.

*Let $z \in V(G) \setminus (A_1 \cup A_2 \cup A_3 \cup X \cup Y)$ be anticomplete to $A_1$. Then $z$ is not mixed on any non-edge with one end in $X \setminus N(b)$ and the other in $Y \setminus N(a)$. In particular, either $x$ is adjacent to $y$, or some $z \in V(G) \setminus (A_1 \cup A_2 \cup A_3 \cup \{x, y\})$ is complete to $\{x, y\}$ and anticomplete to $A_1$.* (4)

Suppose $z$ is mixed on a non-edge $x', y'$ with $x' \in X \setminus N(b)$, and $y' \in Y \setminus N(a)$. From the maximality of the tripod, we may assume that $z$ is anticomplete to $A_2$. Now one of the induced paths $z$-$x'$-$a$-$n_2$-$b$-$y'$ and $z$-$y'$-$b$-$n_2$-$a$-$x'$ is a $P_6$, a contradiction. The second statement follows from assumption 1. This proves (4).

By symmetry, we may assume that $t(n_2) > t(n_3)$, and that $n_2 = n_2(a)$. Thus, there is an induced path $n_2$-$n_3'$-$c$-$n_3$ in $T_1(t(n_2))$. Hence $t(c) < t(n_2)$, and so $a$ is non-adjacent to $c$.

*Vertex $a$ is adjacent to $n_3'$, and $b$ has a neighbor among the set $\{c, n_3'\}$.* (5)

Suppose first that $x$ is adjacent to $y$. If $a$ is non-adjacent to $n_3'$, then $y$-$x$-$a$-$n_2$-$n_3'$-$c$ is a $P_6$, a contradiction. Moreover, if $b$ is anticomplete to $\{c, n_3'\}$, then $x$-$y$-$b$-$n_2$-$n_3'$-$c$ is a $P_6$, a contradiction. So we may assume that $x$ is non-adjacent to $y$, and thus, by the choice of $x$ and $y$, deduce that $X \setminus N(b)$ is anticomplete to $Y \setminus N(a)$.

Now it follows from (4) that every $z \in V(G) \setminus (A_1 \cup A_2 \cup A_3 \cup X \cup Y)$ that is anticomplete to $A_1$ and that has a neighbor in $(X \setminus N(b)) \cup (Y \setminus N(a))$ is already complete to $(X \setminus N(b)) \cup (Y \setminus N(a))$. By assumption 2.(ii), we deduce that $X \setminus N(b) = \{x\}$, and similarly $Y \setminus N(a) = \{y\}$. Moreover, by assumption 4, there exist $x' \in X \cap N(b)$ and $y' \in Y \cap N(a)$ such that $xy'$ and $yx'$ are edges. Now if $a$ is non-adjacent to $n_3'$, then $y$-$x'$-$a$-$n_2$-$n_3'$-$c$ is a $P_6$, and if $b$ is anticomplete to $\{c, n_3'\}$, then $x$-$y'$-$b$-$n_2$-$n_3'$-$c$ is a $P_6$, in both cases a contradiction. This proves (5).

If $b$ is adjacent to $n_3'$, then (a) holds, and thus we may assume the opposite. By (5), $b$ is adjacent to $c$. Since $x$-$a$-$n_3'$-$c$-$b$-$y$ is not a $P_6$, we may deduce that $x$ is adjacent to $y$. Similarly, $X \setminus N(b)$ is complete to $Y \setminus N(a)$.

Let $d = n_1(n_3')$. Then $t(d) \leq t(n_2) < t(a)$, and therefore $a \neq d$. Since $d$-$n_3'$-$a$-$x$-$y$-$b$ is not a $P_6$, we deduce that $d$ is complete to one of $X \setminus N(a)$ and $Y \setminus N(b)$.

By (2), $d$ is not complete to both $X \setminus N(b)$ and $Y \setminus N(a)$. Suppose first that $d$ is complete to $X \setminus N(b)$. Then there is some $y' \in Y \setminus N(a)$ that is non-adjacent to $d$. Since $n_3'$-$d$-$x$-$y'$-$b$-$n_3$ is not a $P_6$, we deduce that $d$ is adjacent to $n_3$. Since $t(d) < t(a)$, $d$ is not complete to $X$, and so there is $x' \in X \cap N(b)$ that is non-adjacent to $d$. Since $x'$-$b$-$c$-$n_3'$-$d$-$x$ is not a $P_6$, $d$ is adjacent to $c$. But $dx, bx'$ is a 2-edge matching between $\{d, b\}$ and $X$, and $\{d, b\}$ is complete to $\{c, n_3'\}$, contrary to 4.4.(a).

This proves that $d$ is not complete to $X \setminus N(b)$, and thus $d$ is complete to $Y \setminus N(a)$ and has a non-neighbor $x' \in X \setminus N(b)$. Suppose that $d$ is non-adjacent to $n_2$. Since $t(n_2(d)) \leq t(d) \leq t(n_2)$, we may deduce that $t(n_2(d)) < t(n_2)$, and $a$ is non-adjacent to $n_2(d)$ (since $n_2 = n_2(a)$). But now $n_2(d)$-$d$-$y$-$x'$-$a$-$n_2$ is a $P_6$, a contradiction. This proves that $d$ is adjacent to $n_2$.

Since $\{a, d\}$ is complete to $\{n_2, n_3'\}$, we deduce that there is no 2-edge matching between $Y$ and $\{a, d\}$, by 4.4.(a). But then $d$ is complete to $Y$, and (b) holds, since $n_2$ is adjacent to $n_3'$. This completes the proof. $\square$

**4.6.** *Let $G'$ be the graph obtained from $G$ by contracting $(A_1, A_2, A_3)$ to a triangle $a_1a_2a_3$. Let $H'$ be an induced subgraph of $G'$ with $a_1 \in V(H')$. Assume that no two non-adjacent neighbors of $a_1$ dominate each other in $H'$. Moreover, assume also that for every $v \in V(H')$, either*

1. *$N_{H'}(v) = X' \cup Y'$, each of $X'$, $Y'$ is stable,*
   - *(i) for every $x \in X'$ and $y \in Y'$, either*
     - *(A) $x$ is adjacent to $y$,*
     - *(B) $x$ has a neighbor in $V(H') \setminus (N_{H'}(v) \cup \{v\})$, or*
     - *(C) $y$ has a neighbor in $V(H') \setminus (N_{H'}(v) \cup \{v\})$;*
   - *(ii) for every $x, x' \in X$ there exists $p \in V(H') \setminus \{v\}$ such that $p$ is non-adjacent to $v$, and $p$ is adjacent to exactly one of $x, x'$;*
   - *(iii)* (1ii) *holds for $Y$ in an analogous way.*
2. *$N_{H'}(v)$ is a triangle, or*
3. *$N_{H'}(v)$ induces a $C_5$.*

*Then either*

*(a) some $a \in A_1$ is complete to $N_H'(a_1) \setminus \{a_2, a_3\}$; or*

*(b) assumption 1 holds, and no vertex of $A_1$ is complete to $N_{H'}(a_1) \setminus \{a_2, a_3\}$, and there exist $a, b \in A_1$, $n_2 \in A_2$, and $n_3 \in A_3$ such that $a$ is complete to $X' \setminus \{a_2, a_3\}$, $b$ is complete to $Y' \setminus \{a_2, a_3\}$, $\{a, b\}$ is complete to $\{n_2, n_3\}$, and $n_2$ is adjacent to $n_3$;*

*(c) assumption 2 or 3 holds, and $G$ contains a non-3-colorable graph with seven or eight vertices; or*

*(d) assumption 2 or 3 holds, there exists a set $A \subseteq A_1$, with $|A| \leq 3$, $n_2 \in A_2$, and $n_3 \in A_3$ such that every vertex of $N_{H'}(a_1)$ has a neighbor in $A$, $A$ is complete to $\{n_2, n_3\}$, and $n_2$ is adjacent to $n_3$.*

*Moreover, suppose $A_2, A_3$ are in $V(H')$. If* (a) *holds, let $H = G|((V(H') \setminus \{a_1\}) \cup \{a\})$. Then $H$ is isomorphic to $H'$. If* (b) *holds, let $H = G|((V(H') \setminus \{a_1\}) \cup \{a, b, n_1, n_2\})$. Then in every coloring of $H$, $a$ and $b$ have the same color. If* (d) *holds, let $H = G|((V(H') \setminus \{a_1\}) \cup A \cup \{n_1, n_2\})$. Then in every coloring of $H$, $A$ is monochromatic.*

*In all cases, $H$ is 3-colorable if and only if $H'$ is.*

*Proof.* Suppose (a) does not hold.

Assume first that assumption 1 holds for $a_1$. Let $X = X' \setminus \{a_2, a_3\}$ and $Y = Y' \setminus \{a_2, a_3\}$. We now quickly check that the assumptions of 4.5 hold for $A_1, X, Y$ (in $G$).

- Every vertex $v \in X \cup Y$ has a neighbor in $A_1$, since every such $v$ is adjacent to $a_1$ in $H'$.
- Assumption 1 of 4.5 follows from assumption 1.(i) of 4.6.
- Assumption 2 holds since there is such a $p$ by assumption 1.(ii) of 4.6. Since $p$ is non-adjacent to $a_1$, we deduce that $p \notin \{a_2, a_3\}$, and so $p \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$, as desired.
- Assumption 3 of 4.5 follows analogously.
- Assumption 4 of 4.5 is seen like this: $N(u) \setminus \{a_1\}$ and $N(v) \setminus \{a_1\}$ are incomparable in $H'$, and $\{u, v\}$ is anticomplete to $\{a_2, a_3\}$ by the maximality of the tripod.

Now 4.6 follows from 4.5.

Next assume that assumption 2 holds for $a_1$, and $N(a_1) = \{x_1, x_2, x_3\}$. We observe that by the maximality of the tripod, $\{x_1, x_2, x_3\} \cap \{a_2, a_3\} = \emptyset$.

Assume first that there exist $b_1, b_2, b_3 \in A_1$ such that $b_i$ is complete to $\{x_j, x_k\}$ (where $\{1, 2, 3\} = \{i, j, k\}$). Since (a) does not hold, $b_i$ is non-adjacent to $x_i$, $i = 1, 2, 3$. If some $n_2 \in A_2$ is complete to $\{b_1, b_2, b_3\}$, then (c) holds. So we may assume that there is a 2-edge matching from $A_2$ to $\{b_1, b_2, b_3\}$, say $n_2b_1, n_2'b_2$. But then $n_2$-$b_1$-$x_2$-$x_1$-$b_2$-$n_2'$ is a $P_6$, a contradiction. So we may assume that no vertex of $A_1$ is adjacent to both $x_1$ and $x_2$. For $i = 1, 2$, let $c_i'$ be the smallest vertex in $A_1$ adjacent to $x_i$ w.r.t. their value of $t$. By 4.5 applied with $X = \{x_1\}$ and $Y = \{x_2\}$, and since no vertex of $A_1$ is adjacent to both $x_1$ and $x_2$, we deduce that there exist a neighbor $c_i$ of $x_i$, and vertices $n_2 \in A_2$ and $n_3 \in A_3$, such that $\{c_1, c_2\}$ is complete to $\{n_2, n_3\}$, and $n_2$ is adjacent to $n_3$. If $x_3$ is adjacent to one of $c_1, c_2$, then (b) holds, so we may supppose this is not the case. Let $c_3$ be a neighbor of $x_3$ in $A$. We may assume that $c_3$ is non-adjacent to $x_1$. Now $c_3$-$x_3$-$x_1$-$c_1$-$n_2$-$c_2$ is not a $P_6$, and so $c_3$ is adjacent to $n_2$. Similarly, $c_3$ is adjacent to $n_3$. But now (c) holds. This finishes the case when assumption 2 holds.

Finally, assume that 3 holds. Let $N_H'(a_1) = \{x_1, \dots, x_5\} = X$, where $x_1$-$x_2$-$\dots$-$x_5$-$x_1$ is a $C_5$. Since $H'|X$ is connected, the maximality of the tripod implies that $a_2, a_3 \notin X$. Let $A$ be a minimum size subset of $A_1$ such that each of $x_1, \dots, x_5$ has a neighbor in $A$. Since every $a \in A$ has a neighbor in $A_2$, we deduce that every $a \in A$ has two non-adjacent neighbors in $X$, due to $P_6$-freeness. We may assume that $|A| > 1$, or (d) holds, and so every $a \in A$ is either a *clone* (i.e., has two non-adjacent or three consecutive neighbors in $X$), a *star* (i.e., has four neighbors in $X$), or a *pyramid* for $G|X$ (i.e., has three neighbors in $X$, one of which is non-adjacent to the other two).

Suppose some $a \in A$ is a clone. We may assume $a$ is adjacent to $x_2$ and $x_5$. If $a$ is mixed on $A_2 \cup A_3$, then, since $T_3$ is connected, there is an induced path $a$-$p$-$q$ where $p, q \in A_2 \cup A_3$. There is also an induced path $a$-$x_2$-$x_3$-$x_4$, so $q$-$p$-$a$-$x_2$-$x_3$-$x_4$ is a $P_6$, a contradiction. So $a$ is complete to $A_2 \cup A_3$. If at most one vertex of $A$ is not a clone and $|A| \leq 3$, then by 4.3 outcome (d) holds. So we may assume that if $|A| \leq 3$, then there are at least two non-clones in $A$.

We claim that $a$ is adjacent to $x_1$. Suppose that this is false, and let $b \in A$ be adjacent to $x_1$. By the minimality of $A$, $b$ is not complete to $\{x_2, x_5\}$. Since $b$ has two non-adjacent neighbors in $X$, by symmetry we may assume that $b$ is adjacent to $x_4$. If $b$ is adjacent to $x_3$, then, by the minimality of $A$, $A = \{a, b\}$ and $b$ is the unique non-clone in $A$, so $b$ is non-adjacent to $x_3$. Now $|A \setminus \{a, b\}| = 1$, and so $b$ is not a clone. Therefore $b$ is adjacent to $x_2$.

By the minimality of $A$, $b$ is non-adjacent to $x_5$. Let $c \in A$ be adjacent to $x_3$. Then $A = \{a, b, c\}$. By the minimality of $A$, $c$ is non-adjacent to $x_5$, and to at least one of $x_1, x_4$. But now $c$ is a clone, and $b$ is the unique non-clone in $A$, a contradiction. So $a$ is adjacent to $x_1$. This implies that $A = \{a, b, c\}$, $b$ is adjacent to $x_4$ but not to $x_5$, $c$ is adjacent to $x_5$ but not $x_4$, neither of $b, c$ is a clone, and no vertex of $A_1$ is complete to $\{x_3, x_4\}$. By 4.5, there exist $b', c' \in A_1$, $n_2 \in A_2$ and $n_3 \in A_3$, such that $b'x_4$ and $c'x_5$ are edges, $n_2$

is adjacent to $n_3$, and $\{b', c'\}$ is complete to $\{n_2, n_3\}$. Now (d) holds. So we may assume that $A$ does not contain a clone.

*If $A = \{a, b\}$ and there exist $x, y, z \in X$ such that $z$-$a$-$x$-$y$-$b$ or $a$-$x$-$y$-$b$-$z$ is an induced path, then* (b) *holds.* (6)

Since $p$-$a$-$x$-$y$-$b$-$q$ is not a $P_6$ for any $p, q \in A_2$, we deduce that either $N(a) \cap A_2 \subseteq N(b) \cap A_2$, or $N(b) \cap A_2 \subseteq N(a) \cap A_2$, and the same holds in $A_3$. Since we may assume (b) does not hold, 4.3 implies that, up to symmetry, there exist $n_2 \in N(a) \cap A_2$ and $n_3 \in N(b) \cap A_3$ such that $a$ is non-adjacent to $n_3$, and $b$ is non-adjacent to $a_2$. Then $n_2$ is adjacent to $n_3$ (or $n_2$-$a$-$x$-$y$-$b$-$n_3$ is a $P_6$). But now $z$-$a$-$n_2$-$n_3$-$b$-$y$ or $z$-$b$-$n_3$-$n_2$-$a$-$x$ is a $P_6$, a contradiction. This proves (6).

Suppose some $a \in A$ is a star, say $a$ is adjacent to $x_1, \ldots, x_4$, and not to $x_5$. Let $b \in A$ be adjacent to $x_5$. Then we know that $A = \{a, b\}$. If $b$ is adjacent to both $x_1$ and $x_4$, then (c) holds, and so we may assume that $b$ is non-adjacent to $x_1$. Since $b$ is not a clone, $b$ is adjacent to $x_2$. If $b$ is adjacent to $x_3$, then (c) holds, so $b$ is non-adjacent to $x_3$; since $b$ is not a clone, $b$ is adjacent to $x_4$. But now (6) holds with $x = x_1$, $y = x_5$ and $z = x_3$. So we may assume that no $a \in A$ is a star, and so every vertex of $A$ is a pyramid.

Let $a \in A$. We may assume that $a$ is adjacent to $x_1, x_3, x_4$ and not to $x_2, x_5$. Let $b \in A$ be adjacent to $x_2$. If $N(b) \cap X = \{x_2, x_4, x_5\}$, then (b) holds by (6) applied with $x = x_3$, $y = x_2$ and $z = x_5$. If $N(b) \cap X = \{x_2, x_3, x_5\}$, then we obtain the previous case by exchanging the roles of $a$ and $b$. So we may assume that $N(b) \cap X = \{x_1, x_2, x_4\}$.

Hence, there exists $c \in A \setminus \{a, b\}$ adjacent to $x_5$ with $N(c) \cap X = \{x_1, x_3, x_5\}$. But now every $x \in X$ has a neighbor in $A \setminus \{a\}$, contrary to the minimality of $A$. This shows how the statement of 4.6 follows from assumption 3, completing the proof. □

**4.7.** *Every graph $H'$ on the list $F_1$-$F_{24}$ satisfies the assumptions of 4.6.*

*Proof.* Since $H'$ is a minimal obstruction to 3-coloring, $H'$ has no dominated vertex, meaning any two neighborhoods of vertices are incomparable. Let $v \in V(H')$. If $N(v)$ is not bipartite, then $v$ contains a triangle or $C_5$, and so $V(H') = \{v\} \cup N(v)$, and assumptions 2 or 3 of 4.6 hold. So $N(v)$ is bipartite with a bipartition $(X, Y)$.

We implemented a straightforward program which we used to verify that assumption 1 of 4.6 indeed holds for all 24 4-critical $P_6$-free graphs from 1.1 where $N(v)$ is bipartite. The source code of this program can be downloaded from [7]. □

**4.8.** *Let $G'$ be obtained from $G$ by contracting $(A_1, A_2, A_3)$ to a triangle $a_1a_2a_3$. Let $H'$ be an induced subgraph of $G'$, with $a_1, a_2 \in V(H')$. For $i = 1, 2$, let $Z_i = N(a_i) \setminus \{a_1, a_2, a_3\}$.*

*Assume that*

1. *no two non-adjacent neighbors of $a_1$ dominate each other, and no two non-adjacent neighbors of $a_2$ dominate each other, and*
2. *$H'|N(a_1)$ and $H'|N(a_2)$ are bipartite.*

*If 4.6.*(a) *holds for $a_1$, let $c_1$ be the vertex $a$ of 4.6.*(a)*, set $A = \{c_1\}$ and $Z = \emptyset$. If 4.6.*(b) *holds for $a_1$, let $a, b, n_2(a_1), n_3(a_1)$ be the vertices as in 4.6.*(b)*. Moreover, set $A = \{a, b\}$, and $Z = \{n_2(a_1), n_3(a_1)\}$.*

*If 4.6.*(a) *holds for $a_2$, let $c_2$ be the vertex $a$ of 4.6.*(a)*, set $C = \{c_2\}$, and $W = \emptyset$. If 4.6.*(b) *holds for $a_2$, let $c, d, n_1(a_2), n_3(a_2)$ be the vertices as in 4.6.*(b)*, set $C = \{c, d\}$, and $W = \{n_1(a_2), n_3(a_2)\}$.*

*Then one of the following holds.*

*(a) Outcome 4.6.*(a) *holds for $a_1$, there is $c \in C$, and an induced path $c_1$-$c'$-$a'$-$c$ in $T_3(t)$ where $t = \max(t(c_1), t(c))$, such that $a'$ is complete to $Z_1$. Or the analog statement holds for $a_2$.*

*(b) There is an edge between $A$ and $C$.*

*(c) In $H'$, there is an induced path $a_1$-$q_1$-$q_2$-$a_2$, and a vertex complete to $\{a_1, q_1, q_2\}$ or to $\{a_2, q_2, q_1\}$.*

*(d) There are vertices $n_1 \in A_1$ and $n_2 \in A_2$, such that $n_1$ is complete to $C$, $n_2$ is complete to $A$, and some vertex $s \in A_3$ is complete to $A \cup \{n_1, n_2\}$ or $C \cup \{n_1, n_2\}$. Moreover, if $\max(|A|, |C|) > 1$, then $|V(H')| \leq 13$.*

*If* (a) *holds, let $A = \{a'\}$. If* (b) *or* (c) *holds, let*

$$H = (H' - \{a_1, a_2\}) \cup A \cup C \cup Z \cup W.$$

*If* (d) *holds, we may assume that $n_1$ is complete to $A$, and put*

$$H = (H' - \{a_1, a_2\}) \cup A \cup C \cup \{n_1, n_2, s\} \cup W.$$

*In all cases, in every 3-coloring of $H$, $A$ and $C$ are monochromatic, and no color appears in both $A$ and $C$. Therefore $H$ is 3-colorable if and only if $H'$ is 3-colorabe.*

*Proof.* We may assume that no vertex of $V(G) \setminus (Z_1 \cup A_2 \cup A_3)$ has a neighbor in $A_1$, and no vertex of $V(G) \setminus (Z_2 \cup A_1 \cup A_3)$ has a neighbor in $A_2$ (otherwise we may delete such vertices from $G$ without changing the hypotheses or the outcomes).

Moreover, we may assume that $A$ is anticomplete to $C$, as otherwise (b) holds. Pick $a \in A$ and $c \in C$. Let $t = \max(t(a), t(c))$, and let $c$-$a'$-$c'$-$a$ be an induced path from $a$ to $c$ in $T_3(t)$. If possible, we choose $a'$ to be complete to $C$, and $c'$ complete to $A$.

Assuming (a) does not hold, we derive the following.

*Vertex $a'$ is not complete to $Z_1$, and $c'$ is not complete to $Z_2$.* (7)

We also make use of the following fact.

*Vertex $c'$ is complete to $A$, and $a'$ to $C$.* (8)

To see this, suppose $c'$ is not complete to $A$. Then $A = \{a, b\}$, and $c'$ is non-adjacent to $b$. By the choice of $c'$, we deduce that $n_2(a_1)$ is non-adjacent to $a'$ (otherwise we may replace $c'$ with $n_2(a_1)$). Now $b$-$n_2(a)$-$a$-$c'$-$a'$-$c$ is a $P_6$, a contradiction. Similarly, $a'$ is complete to $C$. This proves (8).

*Let $p \in Z_1$ be non-adjacent to $a'$. Then $p$ has no neighbor in $V(H') \setminus (\{a_1, a_2, a_3\} \cup Z_1 \cup Z_2)$, and $p$ has a neighbor $q \in Z_1$.* (9)

Since $a_2$ does not dominate $p$, $p$ has a neighbor $q \in H'$ non-adjacent to $a_2$. Then in $G$, $q$ is anticomplete to $A_2$. Let $z \in A$ be adjacent to $p$. If $q$ is not in $Z_1$, then $q$ is anticomplete to $A_1$, and so, by (8), $q$-$p$-$z$-$c'$-$a'$-$c$ is a $P_6$ in $G$, a contradiction. This proves (9).

By (7), (9) and the symmetry between $A_1$ and $A_2$, there exist $p, q \in Z_1$ and $s, t \in Z_2$ such that $pq, st$ are edges, $a'$ is non-adjacent to $p$, and $c'$ is non-adjacent to $s$. Let $r \in A$ be adjacent to $p$, and let $u \in C$ be adjacent to $s$. Since $p$-$r$-$c'$-$a'$-$u$-$s$ is not a $P_6$, we may deduce that $p$ is adjacent to $s$.

Let $D$ be the following $C_6$: $r$-$c'$-$a'$-$u$-$s$-$p$-$r$.

*Vertex $p$ is complete to $A$, and $s$ is complete to $C$.* (10)

Suppose $p$ has a non-neighbor $r' \in A$. Then, since $A$ is anticomplete to $C$, $r'$ is a leaf for $D$, in contradiction to 4.1. Similarly, $s$ is complete to $C$. This proves (10).

By (10), we may assume that $r$ is adjacent to $q$, and $u$ is adjacent to $t$. If $q$ is adjacent to $s$, then (c) holds, which we may assume not to be the case. Similarly, $t$ is non-adjacent to $p$. Since $q, t$ are not hats for $D$, by 4.1, we may deduce that $q$ is adjacent to $a'$, and $t$ to $c'$.

Suppose that $|A| > 1$. Then $a'$ is not complete to $Z_1$. By (3) and (4), $a'$ is complete to $Z_1 \setminus (N(r) \cap N(r'))$. Let $(X_1, Y_1)$ be a bipartition of $Z_1$ such that $p \in X$. We may assume $r'$ is complete to $X_1$, and hence that $r$ is not complete to $X_1$. Thus there is a vertex $p' \in X_1$ such that $a'p, rp'$ is 2-edge matching. If $G$ has 16 vertices, then there is a 3-edge induced path $p$-$f$-$g$-$p'$ in $H' \setminus (\{a_1\} \cup N(a_1))$, and so $a$-$p$-$f$-$g$-$p'$-$r$ is a $P_6$, a contradiction.

Let $d \in A_3$ be adjacent to $a'$. If $d$ is adjacent to $c'$, then, since $d$ is not a hat for $D$, we may deduce that $d$ is adjacent to at least one of $r, u$. Similarly, $d$ is complete to one of $A, C$ and (d) holds. So $d$ is non-adjacent to $c'$. But now $d$-$a'$-$c'$-$t$-$s$-$p$ is a $P_6$, a contradiction. This completes the proof. □

By $W_5$ we denote the graph that is $C_5$ with a center.

**4.9.** *Every $H$ on the list except $K_4$ and $W_5$ satisfies the assumptions of 4.8.*

*Proof.* Let $H$ be a graph on the list. Since $H$ is minimal non-3-colorable, $H$ has no dominated vertices, and so assumption 1 of 4.8 holds. If $H|N(v)$ is not bipartite for some $v \in V(H)$, then $H|N(v)$ contains a triangle or a $C_5$, and so $H = K_4$ or $H = W_5$. □

**4.10.** *Let $G'$ be obtained from $G$ by contracting a maximal tripod $(A_1, A_2, A_3)$ to a triangle $\{a_1, a_2, a_3\}$. Let $H'$ from our list $\{F_1, \dots, F_{24}\}$ be an induced subgraph of $G'$. If $H' = K_4$, assume that $|V(H') \cap \{a_1, a_2, a_3\}| < 3$. Then there exists an induced subgraph $H$ of $G$ that is not 3-colorable with at most $|V(H')| + 9$ vertices if $|V(H')| = 16$ and at most $|V(H')| + 15$ vertices if $|V(H')| \leq 13$.*

*Proof.* We may assume that at least one of $a_1, a_2, a_3$ is in $V(H')$. If $|V(H') \cap \{a_1, a_2, a_3\}| = 1$, we are done using 4.6 and 4.7, so we may assume that $|V(H') \cap \{a_1, a_2, a_3\}| \geq 2$. Note that if $H' = K_4$, every edge is in a triangle, and if $H' = W_5$, then every triangle is in a diamond. Hence, the maximality of $(A_1, A_2, A_3)$ implies that $H' \neq K_4, W_5$.

By 4.9, $H'$ satisfies the assumptions of 4.8. Suppose that either $|V(H') \cap \{a_1, a_2, a_3\}| = 2$, or $|V(H') \cap \{a_1, a_2, a_3\}| = 3$, and one of 4.8.(b), 4.8.(c), 4.8.(d) holds for each pair $a_1a_2, a_2a_3, a_1a_3$. For $i = 1, 2, 3$ let $C_i \subseteq A_i$ be as in 4.6.(a) or 4.6.(b), and let $W_i$ be as in 4.6.(b) (in the notation of 4.8).

Let $Z_i$ be the set of neighbors of $a_i$ in $H' - \{a_1, a_2, a_3\}$. If 4.8.(d) holds for $a_i, a_j$, let $N_k$ be like $\{n_i, n_j, s\}$ in 4.8.(d). Otherwise let $N_k = \emptyset$, and note that in both cases $|N_k| \leq 3$. As usual, we may assume $V(G) = A_1 \cup A_2 \cup A_3 \cup (V(H') \setminus \{a_1, a_2, a_3\})$.

Construct $H$ as in 4.8, modifying $H'$ accordingly for each pair $a_1a_2, a_2a_3, a_1a_3$. Observe that the sets of vertices added in each modification are far from disjoint.

More precisely,

- If 4.6.(a) holds for each of $a_1, a_2, a_3$, then $|V(H)| \leq |V(H')| + 9$, as follows.

  We observe $|C_i| = 1$ and $|W_i| = 0$ for each $i$. Since $H = H' - \{a_1, a_2, a_3\} \cup C_i \cup W_i \cup N_i$, $|V(H)| \leq |V(H')| - 3 + 3 + 9 = |V(H')| + 9$.

- If 4.6.(a) holds for exactly two of $a_1, a_2, a_3$, then $|V(H)| \leq |V(H')| + 6$ if $|V(H')| = 16$, and $|V(H)| \leq |V(H')| + 13$ if $|V(H')| \leq 13$, as follows.

  Assume 4.6.(a) holds for $a_1$ and $a_2$. If $|V(H')| = 16$, then 4.8.(b) or 4.8.(c) happens for $a_2a_3$ and $a_1a_3$, hence $N_1 = N_2 = \emptyset$, $|C_1| = |C_2| = 1$, $|W_1| = |W_2| = 0$, $|N_3| \leq 3$, $|C_3| = |W_3| = 2$, and so $|V(H)| \leq |V(H')| - 3 + 4 + 2 + 3 = |V(H')| + 6$. If $|V(H')| \leq 13$, then $|C_1| = |C_2| = 1$, $|W_1| = |W_2| = 0$, $|C_3| = |W_3| = 2$, and $|N_i| \leq 3$ for any $i$, and hence $|V(H)| \leq |V(H')| - 3 + 4 + 2 + 9 = |V(H')| + 12$.

- If 4.6.(a) holds for exactly one of $a_1, a_2, a_3$, then $|V(H)| \leq |V(H')| + 6$ if $|V(H')| = 16$, and $|V(H)| \leq |V(H')| + 15$ if $|V(H')| \leq 13$, as follows.

  Assume 4.6.(a) holds for $a_1$. If $|V(H')| = 16$, then 4.8.(b) or 4.8.(c) happens for $a_1a_2$, $a_2a_3$ and $a_1a_3$, and hence $N_1 = N_2 = N_3 = \emptyset$, $|C_1| = 1$, $|W_1| = 0$, $|C_2| = |C_3| = |W_2| = |W_3| = 2$, and so $|V(H)| \leq |V(H')| - 3 + 5 + 4 + 0 = |V(H')| + 6$. If $|V(H')| \leq 13$, then $|C_1| = 1$, $|W_1| = 0$, $|C_2| = |C_3| = |W_2| = |W_3| = 2$, $|N_i| \leq 3$ for any $i = 1, 2, 3$, hence $|V(H)| \leq |V(H')| - 3 + 5 + 4 + 9 = |V(H')| + 15$.

- If 4.6.(b) holds for all of $a_1, a_2, a_3$, then $|V(H)| \leq |V(H')| + 9$ if $|V(H')| = 16$, and $|V(H)| \leq |V(H')| + 15$ if $|V(H')| \leq 13$, as follows.

  If $|V(H')| = 16$, then 4.8.(b) or 4.8.(c) happens for $a_1a_2$, $a_2a_3$ and $a_1a_3$, hence $N_1 = N_2 = N_3 = \emptyset$, $|C_1| = |W_1| = |C_2| = |C_3| = |W_2| = |W_3| = 2$, and $|V(H)| \leq |V(H')| - 3 + 6 + 6 + 0 = |V(H')| + 9$. If $|V(H')| \leq 13$, $|C_1| = |W_1| = |C_2| = |C_3| = |W_2| = |W_3| = 2$, and $|N_i| \leq 3$ for any $i$. Moreover, if 4.8.(d) holds for at most two pairs, then $|V(H)| \leq |V(H')| - 3 + 6 + 6 + 6 = |V(H')| + 15$.

  Otherwise 4.8.(d) holds for all three pairs. Considering $a_1a_2$, we may assume that $W_3 = \{n_1, n_2, s_3\}$ and $s_3$ is complete to $C_1$. Thus $W_1$ is not needed since $\{s_3, n_2\}$ is enough to ensure that $C_1$ is monochromatic. Similarly, considering $a_2a_3$, we may assume $W_2$ is not needed. Hence $|V(H)| \leq |V(H')| - 3 + 6 + 2 + 9 = |V(H')| + 14$.

Thus we may assume that $|V(H') \cap \{a_1, a_2, a_3\}| = 3$, and 4.8.(a) holds for at least one of the pairs.

Let us call the outcomes (b), (c), (d) of 4.8 *good*.

*Permuting the indices if necessary, there exist $b_2, b_3 \in A_1$, and $C_2 \subseteq A_2$, $C_3 \subseteq C_3$ such that the following holds.*

- *$\{b_2, b_3\}$ is complete to $Z_1$,*
- *$C_2$ and $C_3$ are as in 4.6.(a) or 4.6.(b),*
- *$b_2$ has a neighbor in $C_2$ and none in $C_3$,* (11)
- *$b_3$ has a neighbor in $C_3$ and none in $C_2$, and*
- *one of the good outcomes holds for the pair $C_2, C_3$.*
- *$b_2$ and $b_3$ have a common neighbor in $A_3$.*

In order to prove (11), we first prove a sufficient condition for (11).

*If there exist $C'_i \subseteq A_i$ as in 4.6.(a) or 4.6.(b) such that there is an edge between $C'_1$ and $C'_2$, and an edge between $C'_2$ and $C'_3$, then (11) holds.* (12)

To see this, apply 4.8 to $C'_1, C'_3$. If one of the good outcomes holds, then a good outcome holds for all three pairs among $C'_1, C'_2, C'_3$, and so we may assume that this is not the case. There is a symmetry between $C'_1$ and $C'_3$, so we may assume that $|C'_1| = 1$ and that there is an induced path $c'_1$-$c''_3$-$c''_1$-$c'_3$ in $T_2$, where $\{c'_1\} = C'_1$, $c'_3 \in C'_3$, and $c''_1$ is complete to $Z_1$. If $c''_1$ has a neighbor in $C_2$, or $c'_1$ has a neighbor in $C_3$, then a good outcome holds for all pairs among $\{c''_1\}, C'_2, C'_3$ or $C'_1, C'_2, C'_3$. Hence, we may assume that this is not the case. Now (11) holds, and this proves (12).

We may assume that outcome 4.8.(a) holds for the pair $C_2, C_3$. By modifying $C_2, C_3$ we may assume that there is an edge between $C_2$ and $C_3$ and outcome 4.8.(b) holds for $(C_2, C_3)$. If a good outcome holds for both $C_1, C_2$, and $C_1, C_3$, then a good outcome holds for all three pairs, so we may assume that this is not the case.

So, assume that outcome (a) holds when 4.8 is applied to $C_1, C_2$. If there is $c_1 \in A_1$ that is complete to $Z_1$ and has a neighbor in $C_2$, then (11) holds by (12). So we may assume that there is a vertex $c'_2 \in A_2$ that is complete to $Z_2$, and an induced path $c_1$-$c'_2$-$c'_1$-$c_2$ in $T_3$, where $c_1 \in C_1$ and $C_2 = \{c_2\}$. If a good outcome holds for $C_1, C_3$, then either (11) holds, or a good out come holds for all three pairs among $C_1, \{c_2\}, C_3$ or $C_1, \{c'_2\}, C_3$.

So, we may assume that 4.8.(a) holds for $C_1, C_3$. By the symmetry between $C_1$ and $C_3$, we may assume that there is $d_1 \in A_1$ and an induced path $c_1$-$c'_3$-$d_1$-$c_3$ where $c_3 \in C_3$, $C_1 = \{c_1\}$, and $d_1$ is complete to $Z_1$. But now there is an edge between $C_3$ and $\{d1\}$, and between $C_3$ and $C_2$, and (11) follows from (12). This proves (11).

If 4.8.(b) or 4.8.(c) holds for the pair $C_2, C_3$, let

$$H = G|((V(H') \setminus \{a_1, a_2, a_3\}) \cup \{b_2, b_3\} \cup C_2 \cup C_3 \cup W_2 \cup W_3),$$

and let

$$H'' = G|((V(H') \setminus \{a_1, a_2, a_3\}) \cup \{b_2, b_3\} \cup C_2 \cup C_3).$$

If 4.8.(d) holds for the pair $C_2, C_3$, let

$$H = G|((V(H') \setminus \{a_1, a_2, a_3\}) \cup \{b_2, b_3\} \cup C_2 \cup C_3 \cup \{n_1, n_2, s\}),$$

and let

$$H'' = G|((V(H') \setminus \{a_1, a_2, a_3\}) \cup \{b_2, b_3\} \cup C_2 \cup C_3).$$

Then $|V(H)| \leq |V(H')| + 7$, and so we may assume that $H$ is 3-colorable.

Let us call a 3-coloring of $H''$ *promising* if $C_2$ is monochromatic, $C_3$ is monocromatic, and no color appears in both of $C_2, C_3$. We observe that by 4.6 and 4.8, every 3-coloring of $H$ gives a promising 3-coloring of $H''$. Since $H'$ is not 3-colorable, in every promising coloring of $H''$ the vertices $b_2$ and $b_3$ receive different colors.

Let $c$ be a 3-coloring of $H$. We may assume that $c(b_i) = i$, $c$ is constantly 1 or 3 on $C_2$, and $c$ is constantly 1 or 2 on $C_3$. Then $c(z) = 1$ for every $z \in Z_1$. If $c$ is 1 on $C_2$, then we recolor $b_2$ with color 3, and get a coloring of $H'$, a contradiction. So we may assume that $c$ is 3 on $C_2$, and $c$ is 2 on $C_3$. If no vertex of $Z_2$ has color 1, we recolor $C_2$ with color 1, and recolor $b_2$ with color 3. We obtain coloring of $H$ with $b_2, b_3$ colored in the same color, a contradiction. So, for some $z_2 \in Z_2$, $c(z_2) = 1$. Similarly, for some $z_3 \in Z_3$, $c(z_3) = 1$.

For $i = 2, 3$ let $Z'_i$ be the set of all vertices $z \in Z_i$ with $c(z_i) = 1$. Then $Z_1 \cup Z'_2 \cup Z'_3$ is a stable set. Let $c_i \in C_i$ be adjacent to $b_i$.

$$Z'_2 \textit{ is anticomplete to } V(G) \setminus (Z_2 \cup A_2). \tag{13}$$

Suppose $p \in V(G) \setminus (Z_2 \cup A_2)$ has a neighbor $z_2 \in Z'_2$. Then $p \notin Z_1$. Let $c'_2 \in C_2$ be adjacent to $z_2$. Suppose first that $b_2$ is non-adjacent to $c'_2$. Then $c'_2 \neq c_2$. Let $n_1 \in A_1$ be complete to $\{c_2, c'_2\}$, a possible choice by b. Now $p$-$z_2$-$c'_2$-$n_1$-$c_2$-$b_2$ is a $P_6$, a contradiction. So $c'_2$ is adjacent to $b_2$. Let $n_2 \in A_2$ be adjacent to $b_2$ and $b_3$ (as in (11), with the roles of $A_2$ and $A_3$ exchanged). Then $p$-$z_2$-$c'_2$-$b_2$-$n_2$-$b_3$ is a $P_6$, again a contradiction. This proves (13).

Now, by (13), we can recolor $H''$ by putting $c'(C_2) = 1$ and $c'(Z'_2) = 3$, $c'(b_2) = 3$, which yields a 3-coloring of $H'$, a contradiction. This completes the proof. □

# 5 Obstructions that are 1-vertex extensions of a tripod

In this section, we prove the following statement.

**5.1.** *Let $G$ be a 4-critical $P_6$-free graph. Assume that there is a tripod $T = (A_1, A_2, A_3)$ in $G$ and some vertex $x$ which has a neighbor in each $A_i$, $i = 1, 2, 3$. Then $|V(G)| \leq 18$.*

To see this, let $G$, $T = (A_1, A_2, A_3)$, and $x$ be as in 5.1. Let $a_1, a_2, a_3$ be the root of $T$. It is clear that $V(G) = V(T) \cup \{x\}$. We call $G$ a *1-vertex extension of a tripod.*

## 5.1 Preparation

We may assume that the ordering $A_1 \cup A_2 \cup A_3 = \{v_1, \dots, v_k\}$ has the following property.

**5.2.** *Let $u \in A_\ell$ and $v \in A_k$ for some $\ell, k \in \{1, 2, 3\}$. Moreover, let $\{\ell, \ell', \ell''\} = \{1, 2, 3\}$ and $\{k, k', k''\} = \{1, 2, 3\}$. Assume that* $\max(t(n_{k'}(v)), t(n_{k''}(v))) < \max(t(n_{\ell'}(u)), t(n_{\ell''}(u)))$. *Then $t(v) < t(u)$.*

Let $b_i$ be the neighbor of $x$ in $A_i$ with $t(b_i)$ maximum, for all $i = 1, 2, 3$. We may assume that $t(b_1) > t(b_2) > t(b_3)$.

**5.3.** *We may assume that $N(x) \cap A_1 = \{b_1\}$ and $N(x) \cap A_i = \{b_i\}$ for some $i \in \{2, 3\}$.*

*Proof.* Since $G|(V(T(t(b_1))) \cup \{x\})$ is 4-chromatic we know that $V(G) = V(T(t(b_1))) \cup \{x\}$. In particular, $N(x) \cap A_1 = \{b_1\}$.

To see the second statement, assume that $|N(x) \cap A_2|, |N(x) \cap A_3| \geq 2$. Suppose for a contradiction that $|N(b_1) \cap A_2|, |N(b_1) \cap A_3| \geq 2$, and let $u$ be the vertex in the set $\{b_2, b_3, n_2(b_1), n_3(b_1)\}$ with $t(u)$ maximum. Then $G - u$ is still 4-chromatic, a contradiction.

So we may assume that $|N(b_1) \cap A_i| = 1$ for some $i \in \{2, 3\}$. Note that $T' = (A_1 \setminus \{b_1\} \cup \{x\}, A_2, A_3)$ is a tripod. Consequently, $b_1$ has neighbors in all three classes of $T'$. Since $|N(b_1) \cap (A_1 \cup \{x\})| = |N(b_1) \cap A_i| = 1$, we are done. □

## 5.2 The enumeration algorithm

Consider the following way of traversing the tripod $T$. Initially, the vertices $b_1, b_2, b_3$ are labeled *active*, and all other vertices are unlabeled. Then, we label the vertices $a_1, a_2, a_3$ as *inactive*. Consequently, if $b_3 = a_3$, say, then $b_3$ is labeled inactive.

Iteratively, pick an active vertex, say $u \in A_i$ with $\{i, j, k\} = \{1, 2, 3\}$. Make $n_j(u)$ and $n_k(u)$ active, unless they are labeled already, whether active or inactive. Then label $u$ as inactive and re-iterate, picking another active vertex, if possible.

**5.4.** *Regardless of which active vertex is picked in the successive steps, this procedure terminates and, moreover, every vertex of $T$ is visited during this procedure.*

*Proof.* Clearly this procedure terminates when there is no active vertex left. Since every vertex is labeled active at most once, this proves the first assertion.

Assume now the procedure has terminated. The latter assertion follows from the fact that, if $W$ is the collection of inactive vertices, $G|W$ is already a tripod. Thus, since $b_1, b_2, b_3 \in W$, $G|(W \cup \{x\})$ is 4-chromatic and so $G|(W \cup \{x\}) = G$, due to the choice of $G$. □

Instead of traversing a given tripod, we use this method to enumerate all possible 4-critical $P_6$-free 1-vertex extensions of a tripod. The idea is to successively generate the possible subgraphs induced by the labeled vertices only. This is done by Algorithm 3. Starting from all relevant graphs on the vertex set $\{x, b_1, b_2, b_3, a_1, a_2, a_3\}$, we iteratively add new vertices, mimicking the iterative labeling procedure mentioned above. The following list contains all of these start graphs.

**5.5.** *We may assume that the graph $G' := G|\{x, b_1, b_2, b_3, a_1, a_2, a_3\}$ has the following properties.*

*(a) If $b_1 = a_1$, then $G = G'$ is $K_4$.*

*(b) If $b_1 \neq a_1$ and $b_2 = a_2$, then $b_3 = a_3$. Moreover,*

$$E(G') \supseteq \{xb_1, xa_2, xa_3, a_1a_2, a_1a_3, a_2a_3\} := F$$
$$E(G') \subseteq F \cup \{b_1a_2, b_1a_3\}.$$

*(c) If $b_1 \neq a_1$, $b_2 \neq a_2$ and $b_3 = a_3$, then*

$$E(G') \supseteq \{xb_1, xb_2, xa_3, a_1a_2, a_1a_3, a_2a_3\} := F$$
$$E(G') \subseteq F \cup \{xa_2, b_1a_2, b_1b_2, b_1a_3, b_2a_1, b_2a_3\}.$$

*(d) If $b_1 \neq a_1$, $b_2 \neq a_2$ and $b_3 \neq a_3$, then*

$$E(G') \supseteq \{xb_1, xb_2, xb_3, a_1a_2, a_1a_3, a_2a_3\} := F$$
$$E(G') \subseteq F \cup \{xa_2, xa_3, b_1a_2, b_1b_2, b_1a_3, b_1b_3, b_2a_1, b_2a_3, b_2b_3, b_3a_1, b_3a_2\}.$$

*Proof.* This follows readily from our assumption $t(b_3) < t(b_2) < t(b_1)$ with 5.2 and 5.3. □

In our algorithm, we do not only consider graphs, but rather tuples containing a graph together with its list of vertex labels and a linear vertex ordering. The algorithm is split into three parts.

- Algorithm 3 initializes all relevant tuples according to 5.5.
- Algorithm 4 is the main procedure, where a certain tuple is extended in all possible relevant ways. This corresponds to a labeling step in our tripod traversal algorithm.
- Algorithm 5 is a subroutine we use to prune tuples we do not need to consider. We call a tuple *prunable* if Algorithm 5 applied to it returns the value *false*.

**Algorithm 3** Generate 4-critical $P_6$-free 1-vertex extension of a tripod

set $V := \{x, b_1, a_1, a_2, a_3\}$ // in this case, $b_2 = a_2$ and $b_3 = a_3$
set $E^{\text{must}} := \{xb_1, xa_2, xa_3, a_1a_2, a_1a_3, a_2a_3\}$
set $E^{\text{may}} := \{b_1a_2, b_1a_3\}$
set $\text{Ord} := (a_3, a_2, a_1, b_1, x)$ and $\text{Act} := \{b_1\}$
set $A_1 := \{a_1, b_1\}$, $A_2 := \{a_2\}$, and $A_3 := \{a_3\}$
**for** each $E \subseteq E^{\text{must}} \cup E^{\text{may}}$ with $E^{\text{must}} \subseteq E$ **do**
    $\text{Expand}(G = (V, E), A_1, A_2, A_3, \text{Ord}, \text{Act})$
**end for**
set $V := \{x, b_1, b_2, a_1, a_2, a_3\}$ // in this case, $b_2 \neq a_2$ and $b_3 = a_3$
set $E^{\text{must}} := \{xb_1, xb_2, xa_3, a_1a_2, a_1a_3, a_2a_3\}$
set $E^{\text{may}} := \{xa_2, b_1b_2, b_1a_2, b_1a_3, b_2a_1, b_2a_3\}$
set $\text{Ord} := (a_3, a_2, a_1, b_2, b_1, x)$ and $\text{Act} := \{b_1, b_2\}$
set $A_1 := \{a_1, b_1\}$, $A_2 := \{a_2, b_2\}$, and $A_3 := \{a_3\}$
**for** each $E \subseteq E^{\text{must}} \cup E^{\text{may}}$ with $E^{\text{must}} \subseteq E$ **do**
    $\text{Expand}(G = (V, E), A_1, A_2, A_3, \text{Ord}, \text{Act})$
**end for**
set $V := \{x, b_1, b_2, b_3, a_1, a_2, a_3\}$ // in this case, $b_2 \neq a_2$ and $b_3 \neq a_3$
set $E^{\text{must}} := \{xb_1, xb_2, xb_3, a_1a_2, a_1a_3, a_2a_3\}$
set $E^{\text{may}} := \{xa_2, xa_3, b_1b_2, b_1b_3, b_2b_3, b_1a_2, b_1a_3, b_2a_1, b_2a_3, b_3a_1, b_3a_2\}$
set $\text{Ord} := (a_3, a_2, a_1, b_3, b_2, b_1, x)$ and $\text{Act} := \{b_1, b_2, b_3\}$
set $A_1 := \{a_1, b_1\}$, $A_2 := \{a_2, b_2\}$, and $A_3 := \{a_3, b_3\}$
**for** each $E \subseteq E^{\text{must}} \cup E^{\text{may}}$ with $E^{\text{must}} \subseteq E$ **do**
    $\text{Expand}(G = (V, E), A_1, A_2, A_3, \text{Ord}, \text{Act})$
**end for**

We now come to the correctness proof of these algorithms.

**5.6.** *Assume that Algorithm 3 terminates and does never generate a tuple whose graph has $k+1$ or $k+2$ vertices, for some $k \geq 4$. Then any 4-critical $P_6$-free graph which is a 1-vertex extension of a tripod has at most $k$ vertices.*

To see this, let $G$ be a 4-critical $P_6$-free graph other than $K_4$ that is a 1-vertex extension of a tripod, with the notation from above. We need the following claim.

**5.7.** *There is a sequence of tuples $\Gamma^i = (G^i = (V^i, E^i), A_1^i, A_2^i, A_3^i, Ord^i, Act^i)$, $i = 0, \ldots, r$, and a way of traversing the tripod $T$ in $r$ steps, in the way described above, for which the following holds, after possibly renaming vertices. Let $V(i)$ be set of all labeled vertices after the $i$-th iteration of the traversal, together with $x$, and let $Act(i)$ be the set of vertices which are active after the $i$-th iteration of the traversal, for $i = 0, \ldots, r$.*

*(a) At some point during the algorithm, Expand$(\Gamma^0)$ is called.*

*(b) During the procedure Expand$(\Gamma^i)$, $\Gamma^{i+1}$ is generated and so Expand$(\Gamma^{i+1})$ is called, for all $i = 0, \ldots, r-1$.*

*(c) The following holds, for all $i = 0, \ldots, r$.*

*(i) $G|V(i) = G^i$, and in particular $A_j \cap V(i) = A_j^i$, for all $j = 1, 2, 3$,*

*(ii) $Act^i = Act(i)$, and*

*(iii) for any two $u, v \in V(i)$ with $t(u) < t(v)$, $u <_{Ord^i} v$.*

*Proof.* Since $G$ is not $K_4$ we may assume that $b_1 \neq a_1$, by 5.5.

If $b_2 = a_2$, then 5.5 implies $b_3 = a_3$, and $\Gamma^0$ is generated by Algorithm 3. Here, $\Gamma^0 = (G^0 = (V^0, E^0), A_1^0, A_2^0, A_3^0, \text{Ord}^0, \text{Act}^0)$ with

**Algorithm 4** Expand(Graph $G = (V, E)$, Set $A_1$, Set $A_2$, Set $A_3$, List Ord, Set Act)

**if** not Feasible($G$, $A_1$, $A_2$, $A_3$, Ord, Act) **then**
  return
**end if**
pick a vertex $u$ from the set Act and let $\{i, j, k\} = \{1, 2, 3\}$ be such that $u \in A_i$
let $u_j$ be the $<_{\mathrm{Ord}}$-minimal neighbor of $u$ in $A_j$, if existent, and let $u_k$ be defined accordingly
  // we write $u <_{\mathrm{Ord}} v$ whenever $u$ appears before $v$ in the list Ord
let $v_j, v_k$ be two entirely new vertices
**for** all ways of inserting $v_j$ and $v_k$ into the list Ord such that
  (a) $a_1 <_{\mathrm{Ord}} v_j, v_k <_{\mathrm{Ord}} u$,
  (b) $v_j <_{\mathrm{Ord}} u_j$, if existent, and $v_k <_{\mathrm{Ord}} u_k$, if existent
 **do**
  put

$$\begin{aligned} E^* := &\{wv_j : w \in A_i \cup A_k \text{ is active}\} \cup \{wv_k : w \in A_i \cup A_j \text{ is active}\} \cup \{xv_j, xv_k\} \cup \{v_j, v_k\} \\ &\cup \{wv_j : w \in A_i \cup A_k \text{ is inactive and has a neighbor } w' \in A_j \text{ with } w' <_{\mathrm{Ord}} v_j\} \\ &\cup \{wv_k : w \in A_i \cup A_k \text{ is inactive and has a neighbor } w' \in A_k \text{ with } w' <_{\mathrm{Ord}} v_k\} \end{aligned}$$

  **for** all subsets $E'$ of $E^*$ **do**
    put $A'_i := A_i$, $A'_j := A_j \cup \{v_j\}$, $A'_k := A_k \cup \{v_k\}$, and $\mathrm{Act}' := (\mathrm{Act} \setminus \{u\}) \cup \{v_j, v_k\}$
    let Ord$'$ be Ord where $v_j$ and $v_k$ are inserted in the position we currently consider
    Expand$((V \cup \{v_j, v_k\}, E \cup E'), A'_1, A'_2, A'_3, \mathrm{Ord}', \mathrm{Act}')$
  **end for**
**end for**
**for** $r = j, k$ **do**
  **if** $u_r$ is existent and $u_r <_{\mathrm{Ord}} u$ **then**
    let $\{r, s\} = \{j, k\}$
    **for** all ways of inserting $v_s$ into the list Ord such that $a_1 <_{\mathrm{Ord}} v_s <_{\mathrm{Ord}} u$ **do**
      put

$$\begin{aligned} E^* := &\{wv_s : w \in A_i \cup A_r \text{ is active}\} \cup \{xv_s\} \\ &\cup \{wv_s : w \in A_i \cup A_r \text{ is inactive and has a neighbor } w' \in A_s \text{ with } w' <_{\mathrm{Ord}} v_s\} \end{aligned}$$

      **for** all subsets $E'$ of $E^*$ **do**
        put $A'_i := A_i$, $A'_s := A_s \cup \{v_s\}$, $A'_r := A_r$, and $\mathrm{Act}' := (\mathrm{Act} \setminus \{u\}) \cup \{v_s\}$
        let Ord$'$ be Ord where $v_s$ is inserted in the position we currently consider
        Expand$((V \cup \{v_s\}, E \cup E'), A'_1, A'_2, A'_3, \mathrm{Ord}', \mathrm{Act}')$
      **end for**
    **end for**
  **end if**
**end for**
**if** both $u_j$ and $u_k$ exist and $u_j, u_k <_{\mathrm{Ord}} u$ **then**
  Expand$(G, A_1, A_2, A_3, \mathrm{Ord}, \mathrm{Act} \setminus \{u\})$
**end if**

- $V^0 = \{a_1, b_1, a_2, a_3, x\}$ and $E^0 = E(G|V^0)$,
- $A^0_1 = \{a_1, b_1\}$, $A^0_2 = \{a_2\}$, and $A^0_3 = \{a_3\}$, and
- $\mathrm{Ord}^0 = (a_3, a_2, a_1, b_1, x)$, and $\mathrm{Act}^0 = \{b_1\}$.

By 5.5, $E^{\mathrm{must}} \subseteq E^0 \subseteq E^{\mathrm{must}} \cup E^{\mathrm{may}}$. The cases when $a_2 \neq b_2$ but $a_3 = b_3$ resp. $a_3 \neq b_3$ are dealt with

**Algorithm 5** Feasible(Graph $G = (V, E)$, Set $A_1$, Set $A_2$, Set $A_3$, List Ord, Set Act)

1: **if** $G$ contains a $P_6$ **then**
2: return **false**
3: **end if**
4: **if** $G$ is not 3-colourable **then**
5: return **false**
6: **end if**
7: **if** $x$ has at least two neighbors in $A_1$ **then**
8: return **false**
9: **end if**
10: **if** $x$ has at least two neighbors in $A_2$ and at least two neighbors in $A_3$ **then**
11: return **false**
12: **end if**
13: **for** any two distinct vertices $u, v \in (V \setminus \{x\})$ with $u <_{\text{Ord}} v$ **do**
14: **if** $u \notin$ Act **then**
15: let $\{i, j, k\} = \{1, 2, 3\}$ be such that $u \in A_i$
16: let $u_j$ be the $<_{\text{Ord}}$-minimal neighbor of $u$ in $A_j$, and let $u_k$ be defined accordingly
17: let $\{i', j', k'\} = \{1, 2, 3\}$ be such that $v \in A_{i'}$
18: let $v_{j'}$ be the $<_{\text{Ord}}$-minimal neighbor of $v$ in $A_{j'}$, if existent, and let $v_{k'}$ be defined accordingly
19: **if** the following hold:
(a) $\{u_j, u_k\} \not\subseteq \{a_1, a_2, a_3\}$,
(b) $v_{j'}$ and $v_{k'}$ both exist, and
(c) $v_{j'}, v_{k'} <_{\text{Ord}} u_r$ for some $r \in \{j, k\}$
**then**
20: return **false**
21: **end if**
22: **end if**
23: **end for**
24: **for** each $u \in (V \setminus \{x\})$ **do**
25: put $W := \{v \in V : v <_{\text{Ord}} u\}$
26: put $B_i := A_i \cap W$ for each $i = 1, 2, 3$
27: **while** there is a vertex $v \in V \setminus (B_1 \cup B_2 \cup B_3 \cup \{u\})$ with neighbors in at least two of $B_1, B_2, B_3$ **do**
28: **if** $v$ has neighbors in all three of $B_1, B_2, B_3$ **then**
29: return **false**
30: **else**
31: put $B_i := B_i \cup \{v\}$, where $B_i$ is the set that $v$ does not have neighbors in
32: **end if**
33: **end while**
34: **end for**
35: return **true**

similarly. This proves (c) for $i = 0$.

For the inductive step assume that for some $s \in \{0, \ldots, r-1\}$ the tuple $\Gamma^s$ has the properties mentioned in (c). We first prove that $\Gamma^{s+1}$ is generated while Expand($\Gamma^s$) is processed, and that $\Gamma^{s+1}$ has the properties mentioned in (c).

First we discuss why Algorithm 5 returns *true* on the input $\Gamma^s$. Clearly $G^s = G|V(s) \neq G$ is 3-colorable and $P_6$-free, and so the if-conditions in lines 4 and 1 both do not apply. Also, the if-conditions in the lines 7 and 10 does not apply to $\Gamma^s$ due to 5.3 applied to $G$ together with (c).(i) in the case $i = s$.

During the steps 13-23, the if-condition in line 19 never applies due to 5.2. To see this, pick two distinct vertices $u, v \in (V^s \setminus \{x\})$ with $u < v$ and $u \notin \mathrm{Act}^s$. Let $\{i, j, k\} = \{1, 2, 3\}$ be such that $u \in A_i^s$, let $u_j$ be the $<_{\mathrm{Ord}^s}$-minimal neighbor of $u$ in $A_j^s$, and let $u_k$ be defined accordingly, let $\{i', j', k'\} = \{1, 2, 3\}$ be such that $v \in A_{i'}^s$, and let $v_{j'}$ be the $<_{\mathrm{Ord}^s}$-minimal neighbor of $v$ in $A_{j'}^s$, if existent, and let $v_{k'}$ be defined accordingly.

Due to property (c).(iii), $t(u) < t(v)$. Since $u \in V^s \setminus \mathrm{Act}^s$, we know that $u \in V(s) \setminus \mathrm{Act}(s)$, by (c).(i). Thus, $n_j(u), n_k(u) \in V(s)$. Moreover, by (c).(i), $n_j(u) = u_j$ and $n_k(u) = u_k$. Now, if $v_{j'}, v_{k'}$ both exist and $v_{j'}, v_{k'} <_{\mathrm{Ord}^s} u_r$ for some $r \in \{j, k\}$, then in particular $t(n_{j'}(v)), t(n_{k'}(v)) < t(n_r(u))$, in contradiction to 5.2.

Finally, $\Gamma^s$ is not pruned in the lines 24-34 since $G - u$ is 3-colorable for every $u \in V$.

Now we argue why $\Gamma^{s+1}$ is constructed and carries the desired properties. If $s = 0$, the case is clear, so we may assume that $s > 0$. Say that, in the procedure Expand($\Gamma^s$), vertex $u$ is picked in line 4 of Algorithm 4. Let us say that $u \in A_i^s$, where $\{i, j, k\} = \{1, 2, 3\}$. In the traversal procedure, $n_j(u)$ and $n_k(u)$ are now visited and made active, in case they are not in $V(s)$ already.

Let us first assume that $n_j(u), n_k(u) \notin V(s)$, and let $v_j, v_k$ be the two entirely new vertices picked in line 6. Due to the definition of tripods, $t(a_1) < t(n_j(u)), t(n_k(u)) < t(u)$, and

$$t(n_\ell(u)) < \min(\{t(w) : w \in N_G(u) \cap A_\ell\} \cup \{\infty\}) \text{ for } \ell = j, k.$$

Consequently, the algorithm considers in line 7 inserting the two new vertices $v_j$ and $v_k$ into $\mathrm{Ord}^s$ such that (c).(iii) holds, where we identify $v_j$ with $n_j(u)$ and $v_k$ with $n_k(u)$. Moreover, $E^*$ in line 8 contains all edges incident to $n_j(u)$ and $n_k(u)$ in $G|V(s)$, due to the definition of $n_j(u)$ and $n_k(u)$. Due to steps 10 and 11, the tuple $\Gamma^{s+1}$ is indeed generated, and Expand($\Gamma^{s+1}$) is called, where

- $G^{s+1} = G|(V(s) \cup \{n_j(u), n_k(u)\}) = G|V(s+1)$, and in particular $A_i^{s+1} = A_i^s = V(s) \cap A_i = V(s+1) \cap A_i$, and $A_\ell^{s+1} = A_\ell^s \cup \{v_\ell = n_\ell(u)\} = V(s+1) \cap A_\ell$ for $\ell = j, k$,
- $\mathrm{Act}^{s+1} = (\mathrm{Act}^{s+1} \setminus \{u\}) \cup \{v_j, v_k\} = (\mathrm{Act}(s) \setminus \{u\}) \cup \{n_j(u), n_k(u)\} = \mathrm{Act}(s+1)$, and
- for any two vertices $u, v \in V(s+1)$ with $t(u) < t(v)$, $u <_{\mathrm{Ord}^{s+1}} v$.

The cases when $n_j(u)$ and/or $n_k(u)$ have been active before are handled analogously. This completes the proof of 5.7. □

Next we derive 5.6.

*Proof of 5.6.* Like above, $\Gamma^r$ is not pruned in step 1 during the procedure of Expand($\Gamma^r$). Since $G^r = G|V(r) = G$, $G$ is indeed generated by the algorithm. As $|V(G^s)| + 2 \geq |V(G^{s+1})|$ for all $s = 0, \ldots, r-1$, $G$ has at most $k$ vertices. □

We implemented this set of algorithms in C with some further optimizations. A crucial detail is how the active vertex is picked in line 4 of Algorithm 4. The following choice seemed to terminate most quickly.

- If the graph which is currently expanded has at most 12 vertices, we pick the Ord-maximal active vertex in line 4.
- If the graph has more than 12 vertices, we pick the active vertex for which the number of non-prunable tuples generated from it is minimum. This is done by trying to extend every active vertex once without iterating any further and counting the number of non-prunable tuples generated.

With this choice, our program does indeed terminate (in about 60 hours) and the largest non-prunable generated graph has 18 vertices. Together with 5.6, we arrive at 5.1. Table 3 shows the number of non-prunable tuples generated by the program.

In order to be sure the algorithm is implemented correctly, we also modified the program so it collects all 4-critical graphs found along the way, similar to line 3 of Algorithm 2. As expected, all 4-critical $P_6$-free 1-vertex extensions of a tripod from our list were found. In the Appendix we describe into more detail how we tested the correctness of our implementation and the source code of the program can be downloaded from [7].

| | | | | | |
|---|---|---|---|---|---|
| $\|V(G)\|$ | 5 | 6 | 7 | 8 | 9 |
| # non-prunable tuples | 3 | 67 | 2,010 | 11,726 | 81,523 |
| $\|V(G)\|$ | 10 | 11 | 12 | 13 | 14 |
| # non-prunable tuples | 388,190 | 1,234,842 | 3,380,785 | 10,669,960 | 16,322,798 |
| $\|V(G)\|$ | 15 | 16 | 17 | 18 | 19, 20 |
| # non-prunable tuples | 137,031 | 49,506 | 2,865 | 330 | 0 |

Table 3: Counts of the number of non-prunable tuples generated by our implementation of Algorithm 3.

# 6 Obstructions up to 28 vertices

In this section we prove the following result.

**6.1.** *Let $G$ be a 4-critical $P_6$-free graph. If $|V(G)| \leq 28$, then $G$ is contained in our list.*

For the proof of this result, we run the enumeration algorithm of Section 1, with the following modifications. In line 1 of Algorithm 2, we do not discard a graph if it contains a diamond, only when it is not $P_6$-free. Moreover, we discard a graph if it contains more than 28 vertices. This procedure terminates exactly with our list (note that the largest graph in our list has 16 vertices). Table 4 shows the number of graphs generated by the algorithm on each relevant number of vertices. This computation took approximately 9 CPU years on a cluster.

| | | | | | | |
|---|---|---|---|---|---|---|
| $\|V(G)\|$ | 5 | 6 | 7 | 8 | 9 | 10 |
| # graphs generated | 1 | 7 | 45 | 253 | 1,385 | 5,402 |
| $\|V(G)\|$ | 11 | 12 | 13 | 14 | 15 | 16 |
| # graphs generated | 12,829 | 24,802 | 36,435 | 41,422 | 42,769 | 46,176 |
| $\|V(G)\|$ | 17 | 18 | 19 | 20 | 21 | 22 |
| # graphs generated | 54,001 | 70,205 | 99,680 | 145,968 | 233,687 | 382,762 |
| $\|V(G)\|$ | 23 | 24 | 25 | 26 | 27 | 28 |
| # graphs generated | 696,462 | 1,430,280 | 3,002,407 | 6,410,184 | 13,703,206 | 30,764,536 |

Table 4: Counts of the number of $P_6$-free graphs generated by our implementation of Algorithm 1 without testing for induced diamonds.

# 7 Proof of 1.1

Let $G$ be a 4-critical $P_6$-free graph. If $G$ is diamond-free, we are done by 3.1. We may thus assume that there is a maximal tripod $T = (A_1, A_2, A_3)$ in $G$ which is not just a triangle.

Suppose that there is some vertex $x \in V(G) \setminus V(T)$ with a neighbor in each $A_i$, $i = 1, 2, 3$. Then $V(G) = V(T) \cup \{x\}$, and so $|V(G)| \leq 18$ by 5.1. By 6.1, $G$ is one of $F_1$-$F_{24}$.

So, we may assume that no vertex has a neighbor in all three classes of $T$. Let $G'$ be the graph obtained by contracting $T$ in $G$. By 2.1, we know that $G'$ is $P_6$-free and not 3-colorable. We may thus pick a 4-critical $P_6$-free subgraph $H$ of $G'$. Inductively, $H$ is one of $F_1$-$F_{24}$. Thus, using 4.10, we see that $|V(G)| \leq 28$. By 6.1, $G$ is one of $F_1$-$F_{24}$.

# 8 $P_7$-free obstructions

This section is devoted to the following unpublished observation by Pokrovskiy [15].

**8.1.** *There are infinitely many 4-critical $P_7$-free graphs.*

In the proof we construct an infinite family of 4-vertex-critical $P_7$-free graphs, i.e., $P_7$-free graphs which are 4-chromatic but every proper induced subgraph is 3-colorable. This means that there is also an infinite number of 4-critical $P_7$-free graphs. Note that, indeed, not all members of our family are 4-critical $P_7$-free.

*Proof of 8.1.* Consider the following construction. For each $r \geq 1$, $G_r$ is a graph defined on the vertex set $v_0, \ldots, v_{3r}$. The graph $G_5$ is shown in Fig. 3. A vertex $v_i$, where $i \in \{0, 1, \ldots, 3r\}$, is adjacent to $v_{i-1}$, $v_{i+1}$, and $v_{i+3j+2}$, for all $j \in \{0, 1, \ldots, r-1\}$. Here and throughout the proof, we consider the indices to be taken modulo $3r + 1$.

First we observe that, up to permuting the colors, there is exacly one 3-coloring of $G_r - v_0$. Indeed, we may w.l.o.g. assume that $v_i$ receives color $i$, for $i = 1, 2, 3$, since $\{v_1, v_2, v_3\}$ forms a triangle in $G_r$. Similarly, $v_4$ receives color 1, $v_5$ receives color 2 and so on. Finally, $v_{3r}$ receives color 3. Since the coloring was forced, our claim is proven.

In particular, $G_r$ is not 3-colorable, since $v_0$ is adjacent to all of $v_1, v_2, v_{3r}$. As the choice of $v_0$ was arbitrary, we know that $G_r$ is 4-vertex-critical.

It remains to prove that $G_r$ is $P_7$-free. Suppose that $P = x_1\text{-}x_2\text{-}\ldots\text{-}x_7$ is an induced $P_7$ in $G_r$. To simplify the argumentation, we assume $G_r$ to be equipped with the proper coloring described above. That is, $v_0$ is has color 4, and, for all $i = 0, \ldots, r-1$ and $j = 1, 2, 3$, the vertex $v_{3i+j}$ is colored with color $j$. Let $X_i$ denote the set of vertices of color $i$, for $i = 1, 2, 3, 4$.

If $r \leq 2$, $|V(G_r)| \leq 7$, and so we are done since obviously $G_2$ is not isomorphic to $P_7$. Therefore, we may assume $r \geq 3$ and, since $G_r$ is vertex-transitive, w.l.o.g. $v_0 \notin V(P)$. Hence, $P$ is an induced $P_7$ in the graph $H := G_r - v_0$, which we consider from now on.

First we suppose that some vertices of $P$ appear consecutively in the ordering $v_1, \ldots, v_{3r}$. That is, w.l.o.g. $x_i = v_j$ and $x_{i+1} = v_{j+1}$ for some $i \in \{1, \ldots, 6\}$ and $j \in \{1, \ldots, 3r-1\}$. Since $P$ is an induced path, we know that neither of $v_{j-1}$ and $v_{j+2}$, if existent, are contained in $P$. Thus, we may assume that $j = 1$, and so $v_3 \notin V(P)$. Recall that $N_H(v_1) \setminus \{v_2\} = X_3$ and $N_H(v_2) \setminus \{v_3\} = X_1$. Thus, $|N_H(x_i) \cap V(P)| \leq 2$ implies $|X_3 \cap V(P)| \leq 1$, and similarly $|N_H(x_{i+1}) \cap V(P)| \leq 2$ implies $|X_1 \cap V(P)| \leq 2$. Therefore, $|X_2 \cap V(P)| = 4$, which means that $x_1, x_3, x_5, x_7 \in X_2$. But this is a contradiction to the fact that $N_H(v_1) \setminus \{v_2\} = X_3$.

Hence, no two vertices of $P$ appear consecutively in the ordering $v_1, \ldots, v_{3r}$. For simplicity, let us say that a vertex $v_i$ is *left of* (*right of*) a vertex $v_j$ if $i < j$ (if $i > j$). We now know the following. Let $x \in V(P)$ be left of $y \in V(P)$. Then $xy \in E$ if and only if $x \in X_1$ and $y \in X_3$, $x \in X_2$ and $y \in X_1$, or $x \in X_3$ and $y \in X_2$. Below we make frequent use of this fact without further reference.

W.l.o.g. $x_1 \in X_1$ and $x_2 \in X_2$. In particular, $x_2$ is left of $x_1$. We now distinguish the possible colorings of the remaining vertices of $P$, obtaining a contradiction in each case.

**Case 1.** $x_3 \in X_1$.

In this case, $x_3$ must be right of $x_2$.

**Case 1.1.** $x_4 \in X_2$.

In this case, $x_4$ is right of $x_1$, and in turn $x_3$ is right of $x_4$. Hence, $x_5$ cannot be in $X_1$, since then it must be right of $x_4$ but left of $x_2$. So, $x_5 \in X_3$, and thus $x_5$ is between $x_2$ and $x_1$.

**Case 1.1.1.** $x_6 \in X_1$.

Then $x_6$ must be left of $x_2$. If $x_7 \in X_2$, it must be left of $x_6$ but right of $x_3$, a contradiction. Otherwise if $x_7 \in X_3$, it must be left of $x_1$ but right of $x_4$, another contradiction.

**Case 1.1.2.** $x_6 \in X_2$.

In this case $x_6$ must be right of $x_3$. If $x_7 \in X_1$, it must be left of $x_4$ but right of $x_6$, a contradiction. Otherwise if $x_7 \in X_3$, it must be left of $x_1$ but right of $x_4$, again a contradiction.

**Case 1.2.** $x_4 \in X_3$.

In this case, $x_4$ is right of $x_3$, and in turn $x_1$ is right of $x_4$.

**Case 1.2.1.** $x_5 \in X_1$.

So, $x_5$ must be left of $x_2$. Hence, $x_6$ cannot be in $X_2$, since then $x_6$ must be left of $x_5$ and right of $x_1$. Thus, $x_6 \in X_3$, which means that $x_6$ is between $x_2$ and $x_3$.

If $x_7 \in X_1$, it must be left of $x_6$ but right of $x_1$, a contradiction. Otherwise if $x_7 \in X_2$, it must be left of $x_4$ but right of $x_1$, another contradiction.

**Case 1.2.2.** $x_5 \in X_2$.

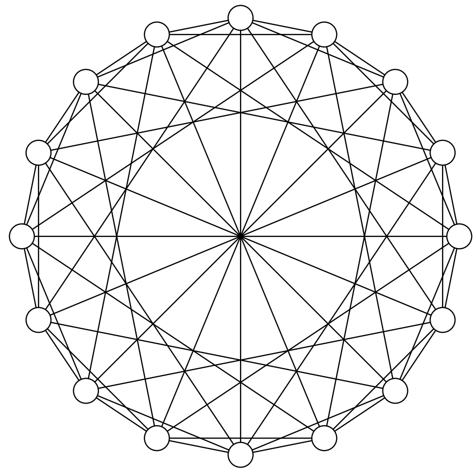

Figure 3: A circular drawing of $G_5$

So, $x_5$ must be right of $x_1$. Clearly $x_6 \notin X_1$, for then it must be left of $x_2$ but right of $x_5$. So, $x_6 \in X_3$, and thus $x_6$ is between $x_2$ and $x_3$.

If $x_7 \in X_1$, it must be left of $x_6$ but right of $x_4$, a contradiction. Otherwise if $x_7 \in X_2$, it must be left of $x_4$ but right of $x_1$, another contradiction.

**Case 2.** $x_3 \in X_3$.

In this case, $x_3$ must be left of $x_2$.

**Case 2.1.** $x_4 \in X_1$.

Then $x_4$ is left of $x_3$, and thus also $x_1$ and $x_2$.

If $x_5 \in X_2$, $x_5$ must be left of $x_4$ but right of $x_1$, a contradiction. So, $x_5 \in X_3$. Then $x_2$ must be between $x_2$ and $x_1$. If $x_6 \in X_2$, it must be right of $x_5$ but left of $x_3$, a contradiction. So, $x_6 \in X_1$, and thus $x_6$ must be between $x_3$ and $x_2$.

If $x_7 \in X_2$, it must be left of $x_6$ but right of $x_1$, a contradiction. Hence, $x_7 \in X_3$. But now $x_7$ must be right of $x_6$ and left of $x_4$, another contradiction.

**Case 2.2.** $x_4 \in X_2$.

Then $x_4$ must be right of $x_1$.

If $x_5 \in X_1$, it must be right of $x_4$ but left of $x_2$, a contradiction. So, $x_5 \in X_3$, and thus $x_5$ is between $x_2$ and $x_1$.

If $x_6 \in X_1$, it must be between $x_3$ and $x_2$. If, moreover, $x_7 \in X_2$, $x_7$ is left of $x_6$ but right of $x_1$, a contradiction. Similarly, if $x_7 \in X_3$, $x_7$ is left of $x_1$ but right of $x_4$, another contradiction.

We thus know $x_6 \in X_2$. But then $x_6$ must be left of $x_3$ and right of $x_5$, a contradiction.

Summing up, $G_r$ is $P_7$-free, and this completes the proof. □

We also modified Algorithm 2 to generate 4-critical $P_7$-free graphs. As one would expect, the number of obstructions is much larger than in the $P_6$-free case. Table 5 contains the counts of all 4-critical and 4-vertex-critical $P_7$-free graphs up to 15 vertices.

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We thank Alexey Pokrovskiy for pointing us to the existence of an infinite family of $P_7$-free obstructions.

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| Vertices | Critical graphs | Vertex-critical graphs |
|---|---|---|
| 4 | 1 | 1 |
| 6 | 1 | 1 |
| 7 | 2 | 7 |
| 8 | 5 | 8 |
| 9 | 21 | 124 |
| 10 | 99 | 2,263 |
| 11 | 212 | 1,771 |
| 12 | 522 | 6,293 |
| 13 | 679 | 15,064 |
| 14 | 368 | 4,521 |
| 15 | 304 | 2,914 |
| $\leq 15$ | 2,214 | 32,967 |

Table 5: Counts of all 4-critical and 4-vertex-critical $P_7$-free graphs up to 15 vertices.

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# Appendix 1: Correctness testing

Since several results obtained in this paper rely on computations, it is very important that the correctness of our programs has been thoroughly verified to minimize the chance of programming errors. In the following subsections we explain how we tested the correctness of our implementations.

Since all of our consistency tests passed, we believe that this is strong evidence for the correctness of our implementations.

## Appendix 1.1: Correctness testing of critical $P_t$-free graph generator

We performed the following consistency tests to verify the correctness of our generator for $k$-critical $P_t$-free graphs (i.e. Algorithm 1). The source code of this program can be downloaded from [6].

- We applied the program to generate critical graphs for cases which were already settled before in the literature and verified that our program indeed obtained the same results. More specifically we verified that our program yielded exactly the same results in the following cases:
  - There are six 4-critical $P_5$-free graphs [2].
  - There are eight 5-critical $(P_5, C_5)$-free graphs [11].
  - The Grötzsch graph is the only 4-critical $(P_6, C_3)$-free graph [16].
  - There are four 4-critical $(P_6, C_4)$-free graphs [10].
- We developed an independent generator for $k$-critical $P_t$-free graphs by starting from the program `geng` [13, 14] (which is a generator for all graphs) and adding pruning routines to it for colorability and $P_t$-freeness. This generator cannot terminate, but we were able to independently verify the following results with it:
  - We executed this program to generate all 4-critical $(P_6, \text{diamond})$-free graphs up to 16 vertices and it indeed yielded the same 6 critical graphs from 3.1.
  - We executed this program to generate all 4-critical and 4-vertex-critical $P_6$-free graphs up to 16 vertices and it indeed yielded the same graphs from 1.1 and Table 1.
  - We executed this program to generate all 4-critical and 4-vertex-critical $P_7$-free graphs up to 13 vertices and it indeed yielded the same graphs from Table 5.
- We modified our program to generate all $P_t$-free graphs and compared it with the known counts of $P_t$-free graphs for $t = 4, 5$ on the On-Line Encyclopedia of Integer Sequences [18] (i.e. sequences A000669 and A078564).

- We modified our program to generate all $k$-colourable graphs and compared it with the known counts of $k$-colourable graphs for $k = 3, 4$ on the On-Line Encyclopedia of Integer Sequences [18] (i.e. sequences A076322 and A076323).
- We determined all $k$-vertex-critical graphs in two independent ways and both methods yielded exactly the same results:
  1. By modifying line 3 of Algorithm 2 so it tests for $k$-vertex-criticality instead of $k$-criticality.
  2. By recursively adding edges in all possible ways to the set of critical graphs (as long as the graphs remain $k$-vertex-critical) and testing if the resulting graphs are $P_t$-free.

### Appendix 1.2: Correctness testing of tripod generator

We performed the following consistency tests to verify the correctness of our generator for 4-critical $P_6$-free 1-vertex extensions of tripods (i.e. Algorithm 3). The source code of this program can be downloaded from [7].

- We wrote a program to test if a graph is a 1-vertex extension of a tripod and applied it to the 24 4-critical $P_6$-free graphs from Theorem 1.1. 11 of those graphs are 1-vertex extensions of a tripod (i.e. $F_1$, $F_2$, $F_4$, $F_6$, $F_7$, $F_9$, $F_{10}$, $F_{17}$, $F_{21}$, $F_{22}$ and $F_{23}$). We verified that our implementation of Algorithm 3 indeed yielded exactly those 11 graphs which are a 1-vertex extension of a tripod (except $K_4$).
- We used Algorithm 1 to generate all 4-critical $P_7$-free graphs up to 14 vertices. There are 1910 such graphs and 595 of them are 1-vertex extensions of a tripod (see Table 6 for details). We modified our implementation of Algorithm 3 to generate 4-critical $P_7$-free 1-vertex extensions of tripods and executed it up to 14 vertices. We verified that this indeed yields exactly those 595 graphs which are a 1-vertex extension of a tripod (except $K_4$).

| Vertices | Critical graphs | 1-vertex extensions |
|---|---|---|
| 4 | 1 | 1 |
| 6 | 1 | 1 |
| 7 | 2 | 1 |
| 8 | 5 | 4 |
| 9 | 21 | 14 |
| 10 | 99 | 56 |
| 11 | 212 | 87 |
| 12 | 522 | 141 |
| 13 | 679 | 196 |
| 14 | 368 | 94 |
| $\leq 14$ | 1,910 | 595 |

Table 6: Counts of 4-critical $P_7$-free graphs up to 14 vertices and the number of those graphs which are 1-vertex extensions of a tripod.

## Appendix 2: Adjacency lists

This section contains the adjacency lists of the 24 4-critical $P_6$-free graphs from Theorem 1.1. The graphs are listed in the same order as in Fig. 1 and 2.

- Graph $F_1$: {0 : 1 2 3; 1 : 0 2 3; 2 : 0 1 3; 2 : 0 1 3}

- Graph $F_2$: {0 : 2 3 5; 1 : 3 4 5; 2 : 0 4 5; 3 : 0 1 5; 4 : 1 2 5; 5 : 0 1 2 3 4}
- Graph $F_3$: {0 : 2 4 5; 1 : 3 5 6; 2 : 0 4 6; 3 : 1 5 6; 4 : 0 2 6; 5 : 0 1 3; 6 : 1 2 3 4}
- Graph $F_4$: {0 : 3 4 5; 1 : 3 5 6; 2 : 4 5 6; 3 : 0 1 4 6; 4 : 0 2 3 6; 5 : 0 1 2; 6 : 1 2 3 4}
- Graph $F_5$: {0 : 3 4 5; 1 : 4 6 7; 2 : 5 6 7; 3 : 0 6 7; 4 : 0 1 5; 5 : 0 2 4; 6 : 1 2 3 7; 7 : 1 2 3 6}
- Graph $F_6$: {0 : 3 5 6; 1 : 4 5 7; 2 : 5 6 7; 3 : 0 6 7; 4 : 1 6 7; 5 : 0 1 2; 6 : 0 2 3 4 7; 7 : 1 2 3 4 6}
- Graph $F_7$: {0 : 3 4 5 7; 1 : 4 5 6; 2 : 5 6 7; 3 : 0 6 7; 4 : 0 1 7; 5 : 0 1 2; 6 : 1 2 3 7; 7 : 0 2 3 4 6}
- Graph $F_8$: {0 : 3 5 7; 1 : 4 7 8; 2 : 5 6 7; 3 : 0 6 8; 4 : 1 7 8; 5 : 0 2 8; 6 : 2 3 8; 7 : 0 1 2 4; 8 : 1 3 4 5 6}
- Graph $F_9$: {0 : 4 5 8; 1 : 4 7 8; 2 : 5 6 8; 3 : 6 7 8; 4 : 0 1 6 8; 5 : 0 2 7; 6 : 2 3 4 8; 7 : 1 3 5; 8 : 0 1 2 3 4 6}
- Graph $F_{10}$: {0 : 4 5 7; 1 : 4 7 8; 2 : 5 6 7; 3 : 6 7 8; 4 : 0 1 6 8; 5 : 0 2 8; 6 : 2 3 4 8; 7 : 0 1 2 3; 8 : 1 3 4 5 6}
- Graph $F_{11}$: {0 : 3 4 5 8; 1 : 4 5 6; 2 : 5 6 7 8; 3 : 0 6 7; 4 : 0 1 7 8; 5 : 0 1 2; 6 : 1 2 3 8; 7 : 2 3 4; 8 : 0 2 4 6}
- Graph $F_{12}$: {0 : 3 6 9; 1 : 4 6 7; 2 : 5 7 8; 3 : 0 6 9; 4 : 1 8 9; 5 : 2 7 8; 6 : 0 1 3 8; 7 : 1 2 5 9; 8 : 2 4 5 6; 9 : 0 3 4 7}
- Graph $F_{13}$: {0 : 4 6 9; 1 : 5 6 8; 2 : 6 8 9; 3 : 7 8 9; 4 : 0 7 8; 5 : 1 7 9; 6 : 0 1 2 7; 7 : 3 4 5 6; 8 : 1 2 3 4 9; 9 : 0 2 3 5 8}
- Graph $F_{14}$: {0 : 4 5 7 9; 1 : 5 6 7; 2 : 6 7 8; 3 : 7 8 9; 4 : 0 6 8; 5 : 0 1 8 9; 6 : 1 2 4 9; 7 : 0 1 2 3; 8 : 2 3 4 5; 9 : 0 3 5 6}
- Graph $F_{15}$: {0 : 4 5 8 9; 1 : 4 7 8 9; 2 : 5 6 8; 3 : 6 7 8; 4 : 0 1 6 9; 5 : 0 2 7; 6 : 2 3 4 9; 7 : 1 3 5; 8 : 0 1 2 3; 9 : 0 1 4 6}
- Graph $F_{16}$: {0 : 5 6 7; 1 : 5 6 9; 2 : 5 8 9; 3 : 6 7 8; 4 : 7 8 9; 5 : 0 1 2 7 8; 6 : 0 1 3 8 9; 7 : 0 3 4 5 9; 8 : 2 3 4 5 6; 9 : 1 2 4 6 7}
- Graph $F_{17}$: {0 : 3 5 6 9; 1 : 4 6 8; 2 : 5 6 7 8 9; 3 : 0 7 8 9; 4 : 1 7 9; 5 : 0 2 7 8; 6 : 0 1 2; 7 : 2 3 4 5; 8 : 1 2 3 5 9; 9 : 0 2 3 4 8}
- Graph $F_{18}$: {0 : 5 6 10; 1 : 5 9 10; 2 : 6 7 10; 3 : 7 8 10; 4 : 8 9 10; 5 : 0 1 7 8; 6 : 0 2 8 9; 7 : 2 3 5 9; 8 : 3 4 5 6; 9 : 1 4 6 7; 10 : 0 1 2 3 4}
- Graph $F_{19}$: {0 : 4 6 7 10; 1 : 5 9 10; 2 : 6 8 9 10; 3 : 7 8 9 10; 4 : 0 8 9; 5 : 1 9 10; 6 : 0 2 7; 7 : 0 3 6; 8 : 2 3 4 10; 9 : 1 2 3 4 5; 10 : 0 1 2 3 5 8}
- Graph $F_{20}$: {0 : 5 10 11; 1 : 6 7 10 11; 2 : 6 9 10 11; 3 : 7 8 10 11; 4 : 8 9 10 11; 5 : 0 10 11; 6 : 1 2 8 10; 7 : 1 3 9; 8 : 3 4 6 11; 9 : 2 4 7; 10 : 0 1 2 3 4 5 6; 11 : 0 1 2 3 4 5 8}
- Graph $F_{21}$: {0 : 4 6 7 10 11; 1 : 5 6 7 8 11; 2 : 6 8 10 11 12; 3 : 7 8 9 10 11 12; 4 : 0 8 9 12; 5 : 1 9 10 11 12; 6 : 0 1 2 9 10 12; 7 : 0 1 3 12; 8 : 1 2 3 4; 9 : 3 4 5 6 11; 10 : 0 2 3 5 6; 11 : 0 1 2 3 5 9; 12 : 2 3 4 5 6 7}
- Graph $F_{22}$: {0 : 4 6 8 9 11 12; 1 : 5 6 7 10 11 12; 2 : 6 7 8 9 11 12; 3 : 9 10 11 12; 4 : 0 7 10 11 12; 5 : 1 8 9 12; 6 : 0 1 2 10; 7 : 1 2 4 9; 8 : 0 2 5 10 11; 9 : 0 2 3 5 7 10; 10 : 1 3 4 6 8 9; 11 : 0 1 2 3 4 8; 12 : 0 1 2 3 4 5}

- Graph $F_{23}$: $\{0 : 4\ 6\ 7\ 9\ 10;\ 1 : 5\ 7\ 8\ 9;\ 2 : 6\ 7\ 9\ 10\ 11;\ 3 : 7\ 8\ 9\ 10\ 11\ 12;\ 4 : 0\ 8\ 9\ 10\ 11\ 12;\ 5 : 1\ 10\ 11\ 12;\ 6 : 0\ 2\ 8\ 11\ 12;\ 7 : 0\ 1\ 2\ 3\ 11\ 12;\ 8 : 1\ 3\ 4\ 6;\ 9 : 0\ 1\ 2\ 3\ 4\ 12;\ 10 : 0\ 2\ 3\ 4\ 5;\ 11 : 2\ 3\ 4\ 5\ 6\ 7;\ 12 : 3\ 4\ 5\ 6\ 7\ 9\}$

- Graph $F_{24}$: $\{0 : 4\ 8\ 13\ 14\ 15;\ 1 : 5\ 8\ 10\ 14\ 15;\ 2 : 6\ 8\ 9\ 10\ 15;\ 3 : 7\ 8\ 9\ 10\ 11;\ 4 : 0\ 9\ 10\ 11\ 12;\ 5 : 1\ 9\ 11\ 12\ 13;\ 6 : 2\ 11\ 12\ 13\ 14;\ 7 : 3\ 12\ 13\ 14\ 15;\ 8 : 0\ 1\ 2\ 3\ 11\ 12\ 13;\ 9 : 2\ 3\ 4\ 5\ 13\ 14\ 15;\ 10 : 1\ 2\ 3\ 4\ 12\ 13\ 14;\ 11 : 3\ 4\ 5\ 6\ 8\ 14\ 15;\ 12 : 4\ 5\ 6\ 7\ 8\ 10\ 15;\ 13 : 0\ 5\ 6\ 7\ 8\ 9\ 10;\ 14 : 0\ 1\ 6\ 7\ 9\ 10\ 11;\ 15 : 0\ 1\ 2\ 7\ 9\ 11\ 12\}$