A Characterization of Line Graphs that are Squares of Graphs

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February 26, 2014

Abstract

The square of a graph $G$, denoted $G^2$, is the graph obtained from $G$ by putting an edge between two distinct vertices whenever their distance in $G$ is at most 2. Motwani and Sudan proved that it is NP-complete to decide whether a given graph is the square of some graph. In this paper we give a characterization of line graphs that are squares of graphs, and show that if a line graph is a square, then it is a square of a bipartite graph. As a consequence, we obtain a linear time algorithm for deciding whether a given line graph is the square of some graph.

Keywords: the square of a graph; line graph; linear time algorithm

Mathematics Subject Classification (2010): 05C12, 05C75, 05C76

1 Introduction

Given two graphs $G$ and $H$ and a positive integer $k$, we say that $G$ is the $k$-th power of $H$ (and denote this by $G = H^k$) if the vertex sets of $G$ and $H$ coincide and two distinct vertices are adjacent in $G$ if and only if they are at distance at most $k$ in $H$. The graph $H$ is then called a $k$-th root of $G$. In the case $k = 2$, we say that $G$ is the square of $H$ and $H$ is the square root of $G$. We will say that a graph $G$ is a square graph if it admits a square root.

Graph powers are basic graph transformations with a number of results about their properties in the literature [1, 5, 8, 11, 12, 13, 14, 15, 16]. Motwani and Sudan proved in 1994 that it is NP-complete to decide whether a given graph is the square of some graph [19]. In 2006, Lau [11] proved that determining whether a given graph is the cube

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of some bipartite graph is \( \text{NP} \)-complete. In the same paper, he conjectured that for every fixed \( k \geq 2 \), recognizing \( k \)-th powers of graphs is \( \text{NP} \)-complete and that for every fixed \( k \geq 3 \), recognizing \( k \)-th powers of bipartite graphs is \( \text{NP} \)-complete [11]. Both conjectures were proved by Le and Nguyen [13].

It is also \( \text{NP} \)-complete to determine if a given graph has a square root that is either chordal [12], split [12], of girth four [8], or of girth five [7]. On the other hand, there are polynomial time algorithms for computing a square root that is either a tree [11, 17], a bipartite graph [11], a proper interval graph [12], a block graph [14], a strongly chordal split graph [15], or a graph of girth at least six [8]. Several optimization problems are \( \text{NP} \)-complete for powers of graphs [17].

The complexity of the recognition of square graphs varies when the input graph is restricted to particular graph classes. On the one hand, the problem remains \( \text{NP} \)-complete for chordal graphs [12]. On the other hand, the problem is polynomial for planar graphs [17] and for trivially perfect graphs [18].

In this paper, we continue this line of research, focusing on line graphs. We characterize the line graphs that are the square of some graph. We show that the condition that a line graph \( L \) is the square of a graph can be stated in terms of any of its line roots, as follows: \( L \) is the square of a graph if and only if every connected component of each of its line roots can be obtained from a complete graph by attaching, in an arbitrary manner, pendant vertices to all but at least one vertex. (See Theorem 1 in Section 3.) Graphs obtained this way from complete graphs are called porcupines. We give a characterization of line graphs of porcupines in Theorem 2. Our characterization of line graphs that are squares of graphs also shows that line graphs with a square root have a bipartite square root, and can be used to develop a linear time algorithm for computing a (bipartite) square root (if one exists) of a given line graph.

2 Preliminaries: line graphs

All graphs in this paper will be finite, simple and undirected. Let \( G \) be a graph. The line graph of \( G \), denoted \( L(G) \), has the set \( E(G) \) as its vertex set and two distinct vertices \( e, f \in V(L(G)) \) are adjacent if and only if they share a common vertex in \( G \). We will say that \( G \) is a line root of \( L(G) \). Line graphs have received an enormous amount of interest in graph theory literature. There are many papers discussing the structural and algorithmic properties of the class of line graphs. One of the classic results in this context is the characterization of line graphs in terms of nine forbidden induced subgraphs given by Beineke [4]. For our results, we need only two of these forbidden induced subgraphs. By \( K_n^- \) we denote the graph obtained from the complete graph \( K_n \) by removing an arbitrary edge. By \( P_n \) (resp. \( C_n \)) we denote the chordless path (resp. cycle) on \( n \) vertices.

Proposition 1. A line graph does not contain \( K_5^- \) or \( P_6^2 \) as induced subgraph (see Fig. 1).

Proof. This follows from the fact that \( K_5^- \) and \( P_6^2 \) are contained in the list of forbidden induced subgraphs given by Beineke [4].
3 Results

In this section, we state the main results of this paper. Proofs will be given in Section 4.

Our first result characterizes line graphs that are squares. To state this characterization, we need the notion of porcupines. A porcupine is a graph $G$ obtained from a complete graph by attaching, in an arbitrary manner, pendant vertices to all but at least one vertex of the complete graph. That is, there is a partition of $V(G)$ into vertex sets $U$ and $V$ such that $U$ is a non-empty clique and every member of $V$ is a pendant vertex in $G$. Moreover, there is at least one vertex in $U$ that does not have a neighbor in $V$. Notice that this definition of a porcupine is slightly more restrictive than the one in [9], which also allows that every vertex in $U$ has a neighbor in $V$.

In particular, every complete graph is a porcupine, and the only trees that are porcupines are stars. An example of a porcupine is displayed in Fig. 2.

![Figure 2: A porcupine $P$ (left) and its line graph $L(P)$ (right).](image)

**Theorem 1.** For every connected line graph $L$, the following assertions are equivalent.

1. $L$ is a square graph.
2. $L$ is a square graph with a bipartite square root.
3. Every line root of $L$ is a porcupine. (Equivalently: There exists a line root of $L$ that is a porcupine.)
Corollary 1. For a given line graph $L$, it can be decided in linear time whether $L$ is the square of some graph. Moreover, if $L$ is the square of some graph, a bipartite root of $L$ can be computed in linear time.

The statement of Theorem 1 is illustrated with an example given in Fig. 2 and Fig. 3. Fig. 2 shows a porcupine and its line graph, and a bipartite square root of this line graph is displayed in Fig. 3.

It is known that a graph $L$ is a line graph if and only if the edges of $L$ can be partitioned into complete subgraphs in such a way that no vertex belongs to more than two of the subgraphs (cf. Theorem 7.1.8 in [5]). An analogue of this characterization is given in our next result, characterizing the line graphs of porcupines. (Note that these graphs are, as shown in Theorem 1, exactly the line graphs for which a square root exists.)

**Theorem 2.** Let $L$ be a connected graph. Then, $L$ is the line graph of a porcupine if and only if there is a collection $C_1, \ldots, C_n$ of $n \geq 1$ (not necessarily distinct) cliques in $L$ with the following properties:

(a) The cliques together cover all vertices and edges of $L$.

(b) At least one of the cliques consists of exactly $n - 1$ vertices.

(c) Every vertex of $L$ belongs to at most two distinct cliques.

(d) Every two distinct cliques $C_i, C_j$ have exactly one vertex in common.

### 4 Proofs

We prove our main result, Theorem 1, using a sequence of lemmas. First, using Proposition 1, we show that every connected line graph that is a square graph is of diameter at most 2. The *distance* between two vertices $x, y$ in a connected graph $G$, denoted by $\text{dist}_G(x,y)$, is the length of a shortest path from $x$ to $y$ in $G$. The *diameter* of $G$ is the maximum distance between two vertices of $G$. 

\[ \]
Lemma 1. Let $L$ be a connected line graph. If $L$ is a square graph, then the diameter of $L$ is bounded by 2.

Proof. Let $L$ be a connected line graph with square root $H$. We show that the diameter of $H$ is bounded by 4. Then, the diameter of $L = H^2$ is clearly bounded by 2.

Let $x, y \in V(H)$ be arbitrary and let $P = (x = v_0, v_1, \ldots, v_k = y)$ be a shortest path between $x$ and $y$. Suppose that $k \geq 5$, that is, the distance between $x$ and $y$ is at least 5. Then $L[\{v_0, v_1, \ldots, v_3\}]$ is isomorphic to $P_6^2$, a contradiction to Proposition 1. This completes the proof.

In the next lemma we prove that every line graph that is a square is the line graph of a block graph. For two graphs $F$ and $G$, we say that a graph $G$ is $F$-free if no induced subgraph of $G$ is isomorphic to $F$. A graph $G$ is called chordal if every cycle of $G$ on 4 or more vertices has a chord. Equivalently, $G$ is $C_{k+4}$-free for every $k \geq 0$. A graph is called a block graph if every 2-connected component of it is a clique. The proof of the lemma will rely on the characterization of block graphs in terms of forbidden induced subgraphs.

Proposition 2 ([6], see also [14]). A graph is a block graph if and only if it is chordal and $K_4^-$-free.

Lemma 2. Let $L$ be the line graph of a graph $G$. If $L$ is a square graph, then $G$ is a block graph.

Proof. Let $L = L(G)$ be a line square graph with square root $H$. Due to Proposition 2, it suffices to show that $G$ is chordal and $K_4^-$-free. First we show that $G$ is chordal.

Suppose $C = (v_0, v_1, \ldots, v_k = v_0)$ for $k \geq 4$ is an induced cycle in $G$. Let $e_i$ denote the corresponding edge $v_iv_{i+1}$ for $0 \leq i \leq k-1$. Then $(e_0, e_1, \ldots, e_k = e_0)$ for $k \geq 4$ is an induced cycle in $L(G)$. In particular, $1 \leq \text{dist}_H(e_i, e_{i+1}) \leq 2$, for all $0 \leq i \leq k-1$.

Case 1: Suppose two members of the set $F := \{e_i : 0 \leq i \leq k-1\}$, say $e_i$ and $e_j$ with $i < j$, are adjacent in $H$. Since $F$ induces a cycle in $H^2$, this pair of vertices must be consecutive. Without loss of generality we may assume that $i = 0$ and $j = 1$, that is, $\text{dist}_H(e_0, e_1) = 1$. Since $e_0$ and $e_2$ are non-adjacent in $H^2$, their distance in $H$ must not be smaller than 3. On the other hand, since $\text{dist}_H(e_1, e_2) \leq 2$ it follows that $\text{dist}_H(e_0, e_2) \leq 3$ and consequently $\text{dist}_H(e_0, e_2) = 3$ and $\text{dist}_H(e_1, e_2) = 2$ (see Fig. 4). Let $(e_1, e^*_1, e_2)$ be a shortest path connecting $e_1$ and $e_2$ in $H$. That makes $e^*$ an edge in $G$ adjacent to at least three edges of $C$ in $G$, namely $e_0, e_1$ and $e_2$. This implies either a parallel edge, or a chord in $C$. Both possibilities lead to a contradiction, since we are only allowing simple graphs and $C$ is an induced cycle by definition.

Case 2: Now let none of the edges $e_i e_{i+1}$ be present in $H$, for all $0 \leq i \leq k-1$. Since $e_{i-1}, e_i$ are adjacent in $L(G)$, there has to be a path $(e_{i-1}, e^*_i, e_i)$ in $H$, for all $1 \leq i \leq k$. We set $e^*_0 := e^*_k$. These $e^*_i$, $1 \leq i \leq k$, have to be pairwise distinct: If $e^*_i = e^*_j$ for some $1 \leq i < j \leq k$, then the set \{e_{i-1}, e_i, e_{j-1}, e_j\} would form a clique of at least three vertices in $H^2$, contrary to the fact that $C$ is an induced cycle of order at least 4 in $L = H^2$. Thus, $1 \leq \text{dist}_H(e^*_i, e^*_i) \leq 2$ for all $0 \leq i \leq k-1$, which makes $e^*_i$ and $e^*_i$ adjacent edges in
which makes $e$ a contradiction.

Figure 4: The subgraph $C$ of $G$ for $k = 4$, in the situation of case 1. The white vertices belong to $G$, the black vertices belong to $H$. The dashed lines are edges of $H$.

$G$. Every $e_i^*$ must contain at most one vertex of $C$ and has to be adjacent to $e_{i-1}$, $e_i$, and $e_{i+1}$, for all $1 \leq i \leq k$. Therefore, all $e_i^*$ contain a common vertex, say $v^* \notin C$.

Case 2.1: Suppose $e_1^*$ and $e_3^*$ are neighbors in $H$ (see Fig. 5). Then $\text{dist}_H(e_1^*, e_0) = \text{dist}_H(e_1^*, e_1) = 1$ and $\text{dist}_H(e_1^*, e_2), \text{dist}_H(e_1^*, e_3) \leq 2$. In other words, $e_1^*$ is adjacent to at least four distinct edges of $C$ in $G$ and therefore has to be a chord of $C$. This is a contradiction.

Case 2.2: Suppose that $e_1^*$ and $e_3^*$ are non-adjacent in $H$. Let $(e_1^*, e^*, e_3^*)$ be a shortest path from $e_1^*$ to $e_3^*$ in $H$. Then $\text{dist}_H(e^*, e_0), \text{dist}_H(e^*, e_1), \text{dist}_H(e^*, e_2), \text{dist}_H(e^*, e_3) \leq 2$, which makes $e^*$ adjacent to at least four distinct edges of $C$ in $G$ (see Fig. 6). Again, this is a contradiction.

It remains to show that $G$ is $K_4^-$-free.

Suppose that $G$ contains a $K_4^-$ induced by vertex set $D = \{v_0, v_1, \ldots, v_4 = v_0\}$, with edge set $\{v_0v_1, v_1v_2, v_2v_3, v_3v_4,v_1v_3\}$. Let us denote by $e_i$ the edge $v_iv_{i+1}$, for all $0 \leq i \leq 3$, and by $e_{\text{chord}}$ the edge $v_1v_3$. Also, we set $e_4 := e_0$. Then $(e_0, e_1, e_2, e_3, e_4 = e_0)$ is an induced cycle in $L(G)$. Note that $\text{dist}_H(e_i, e_{i+1}) \leq 2$ and $\text{dist}_H(e_{\text{chord}}, e_i) \leq 2$ for all $0 \leq i \leq 3$, but $\text{dist}_H(e_i, e_{i+2}) > 2$ for $i = 0, 1$.

Up to symmetry, it suffices to consider the following two cases: (1) $\text{dist}_H(e_1, e_2) = 1$ and (2) $\text{dist}_H(e_i, e_{i+1}) = 2$ for all $0 \leq i \leq 3$.

Case 1: Suppose that $\text{dist}_H(e_1, e_2) = 1$ (see Fig. 7). Since $e_0 \cap e_2 = e_1 \cap e_3 = \emptyset$, we
have $\text{dist}_H(e_0, e_1) = \text{dist}_H(e_2, e_3) = 2$. Let $(e_0, e_1^*, e_1)$ and $(e_2, e_3^*, e_3)$ be shortest paths in $H$. By assumption, $\text{dist}_H(e_1, e_2) = 1$, and so $e_1^*$ is adjacent to $e_1$, $e_2$ and $e_3$ in $G$. Therefore $e_3^* = e_{\text{chord}}$. The same holds for $e_1^*$. That makes $e_{\text{chord}}$ neighbor to all $e_j$, $0 \leq j \leq 3$ in $H$. In particular, $e_1$ is adjacent to $e_3$ in $G$, which is a contradiction.

**Case 2:** Suppose that $\text{dist}_H(e_i, e_{i+1}) = 2$ for all $0 \leq i \leq 3$. Let $(e_i, e_i^*, e_{i+1})$ denote a shortest path connecting $e_i$ and $e_{i+1}$ in $H$, for all $0 \leq i \leq 3$. We set $e_0^* := e_1^*$. All edges $e_i^*$, for $1 \leq i \leq 4$ have to be pairwise distinct: If $e_i^* = e_j^*$ for some $1 \leq i < j \leq 4$, then the set $\{e_{i-1}, e_i, e_{j-1}, e_j\}$ would form a clique of at least three vertices in $H^2$, contrary to the fact that $(e_0, e_1, e_2, e_3, e_4 = e_0)$ is an induced cycle in $L = H^2$.

Thus, $1 \leq \text{dist}_H(e_i^*, e_{i+1}^*) \leq 2$ for all $0 \leq i \leq 3$. In particular, $e_i^*$ and $e_{i+1}^*$ are adjacent vertices of $H^2$, for all $0 \leq i \leq 3$, and thus adjacent as edges in $G$. Due to $e_0^* \neq e_2^*$, we can without loss of generality assume that $e_2^* \neq e_{\text{chord}}$. Since $v_2$ is not adjacent to $e_{\text{chord}}$, $e_2^*$ has to contain $v_2$ and some vertex $v^* \notin D$. The edges $e_1^*$ and $e_3^*$ have to share one of these vertices with $e_2^*$ and, thus, cannot be identical to $e_{\text{chord}}$. Hence, $e_1^* = (v_i, v^*)$ for all $1 \leq i \leq 3$. This implies $\text{dist}_H(e_i^*, e_j^*) \leq 2$ for all $1 \leq i < j \leq 3$. Moreover, $\text{dist}_H(e_i^*, e_j^*) = 1$ for some $1 \leq i < j \leq 3$ would imply $e_i^* = e_j^* = e_{\text{chord}}$ ($e_i^*$ as well as $e_j^*$ would have to be adjacent to at least three pairwise distinct edges of $D$ in $G$), which is a contradiction to all $e_i^*$ being pairwise distinct. Hence, $\text{dist}_H(e_i^*, e_j^*) = 2$ for all $1 \leq i < j \leq 3$.

**Case 2.1:** Suppose $e_0^* = e_{\text{chord}}$. Let $(e_1^*, e^*, e_3^*)$ be a shortest path in $H$ (see Fig. 8). Then $e^*$ is adjacent to four pairwise distinct edges of $D$ and, thus, $e^* = e_{\text{chord}}$. But then $e_3^*$ is adjacent to $e_0$ in $G$, since both are adjacent to $e^* = e_{\text{chord}}$ in $H$. This is a contradiction to the fact that $e_0$ and $e_3$ do not share a common vertex in $G$. 

![Figure 6: The subgraph $C$ of $G$ for $k = 4$, in the situation of case 2.2.](image)

![Figure 7: The subgraph $D$ of $G$, in the situation of case 1.](image)
Case 2.2: Now assume $e_0^* \neq e_{\text{chord}} \neq e_2^*$. In this case, $e_0^* = v_0v^*$ and $e_2^* = v_2v^*$. Note that $e_0^*$ and $e_2^*$ are non-adjacent in $H$ since otherwise $e_0^*$ would be adjacent in $H^2$ to four pairwise distinct edges of $D$, which is impossible due to the assumption $e_0^* \neq e_{\text{chord}}$ and the fact that $D$ induces a $K_4^-$ in $G$. Hence, $\text{dist}_{\text{H}}(e_0^*, e_2^*) = 2$. Let $(e_0^*, e^*, e_2^*)$ be a shortest path in $H$. Again $e^*$ has to be adjacent to four pairwise distinct edges of $D$ in $G$ and thus $e^* = e_{\text{chord}}$. But $e_{\text{chord}}$ is adjacent neither to $e_0^*$ nor to $e_2^*$ in $G$, which is a contradiction to $L$ being the line graph of $G$. This completes the proof.

In our next lemma, we show that the only possible connected block graphs whose line graph is a square graph can be found among porcupines. Recall that a porcupine is a graph obtained from a complete graph $K$ by attaching, in an arbitrary manner, pendant vertices to some but not all vertices of $K$.

**Lemma 3.** Let $G$ be a connected block graph. If $L(G)$ is a square graph, then $G$ is a porcupine.

**Proof.** Let $G$ be a connected block graph such that $L(G)$ is a square graph. We may assume, without loss of generality, that $G$ has at least three vertices, since otherwise $G$ is complete and hence a porcupine.

By Lemma 1, the diameter of $L(G)$ is at most 2. It follows that no two vertices in $G$ can have distance 4 or greater.

Recall that block graphs are *distance-hereditary*, that is, every chordless path is a shortest path between its endpoints [3, 10]. Thus, $G$ can not contain $P_5$ as induced subgraph, otherwise the two end-vertices of $P_5$ would have distance 4 in $G$. By chordality, $G$ is also $C_5$-free. Hence, $G$ does not contain $P_5$ or $C_5$ as induced subgraph. As shown by Bacsó and Tuza [2], every $(P_5, C_5)$-free graph has a *dominating clique*, that is, a clique that is also a dominating set. Hence, $G$ has a dominating clique. That is, every vertex of $G$ not contained in this clique has at least one neighbor in it.

**Claim:** $G$ contains a dominating clique of at least two vertices such that every vertex outside the clique is a pendant vertex.

If $G$ is a tree, the claim follows from the assumption that $G$ has at least three vertices. Indeed, any dominating clique (whose existence is guaranteed above) can be extended.
(if necessary) to a clique of size two, and the cycle-freeness guarantees that every vertex outside the clique is a pendant vertex.

Suppose now that \( G \) is not a tree. Since \( G \) is chordal but not a tree, it contains a cycle of order 3. Let \( C \) be a maximal clique in \( G \) of order at least 3. Suppose that \( G \) has another maximal clique of size at least three, say \( C' \). Since \( G \) is a block graph, \( |C \cap C'| \leq 1 \). If \( |C \cap C'| = 1 \), say \( C \cap C' = \{x\} \), \( x \) is a cut-vertex of \( G \), since \( G \) is a block graph. If \( |C \cap C'| = 0 \), there exists a cut-vertex \( x \) of \( G \) such that in \( G - x \), \( C \setminus \{x\} \) and \( C' \setminus \{x\} \) are contained in distinct connected components (again since \( G \) is a block graph). In both cases, there exist edges \( e \in E(G[C \setminus \{x\}]) \) and \( f \in E(G[C' \setminus \{x\}]) \). But then the distance of \( e \) and \( f \) in \( L(G) \) is at least 3, a contradiction to Lemma 1. Hence, \( C \) is the only maximal clique of \( G \) of order at least 3. Using a similar argumentation like above, it is easy to prove that \( C \) is a dominating clique of \( G \). Suppose that there is a vertex \( x \) outside \( C \) of degree 2 or greater. Since \( G \) is a block graph, \( x \) can have at most one neighbor in \( C \), say \( v \). Thus, there is a vertex \( y \in N(x) \setminus C \). This vertex must have a neighbor in \( C \), since \( C \) is a dominating clique. As \( G \) is a block graph, this neighbor must be \( v \). But then \( \{x, y, v\} \) are contained in a maximal clique of size at least 3 different from \( C \), a contradiction. Hence, every vertex outside \( C \) must be of degree 1, and, since \( C \) is dominating, its neighbor belongs to \( C \). This completes the proof of the claim.

The above claim implies that there is a partition of \( V(G) \) into vertex sets \( U \) and \( V \) such that \( U \) is a maximal clique of at order at least 2 and every member of \( V \) is a pendant vertex in \( G \) whose unique neighbor is in \( U \). It remains to show that there exists one vertex in \( U \) that does not have a neighbor in \( V \).

Suppose, for a contradiction, that every member of \( U \) has a neighbor in \( V \). Let \( H \) be a square root of \( L(G) \). The edges of \( G \), and hence vertices of \( L(G) \) and of \( H \), can be partitioned into disjoint sets
\[
E(G) = E_U \cup \bigcup_{u \in U} E_u
\]
where
\[
E_U = \{uu' : u, u' \in U \text{ and } u \neq u'\} \\
E_u = \{uv : v \in N_G(u) \cap V\}, \text{ for all } u \in U.
\]
By assumption, the sets \( E_u \) (for \( u \in U \)) are all non-empty. Notice that for every two distinct vertices \( u \) and \( u' \) from \( U \), every two edges \( e \in E_u \) and \( f \in E_{u'} \) are disjoint. This implies that in the graph \( L(G) \), and consequently in \( H \), there are no edges connecting vertices from two distinct sets \( E_u \) and \( E_{u'} \). Since \( G \) is connected, so is \( H \). Therefore, since there exist at least two distinct sets \( E_u \) and \( E_{u'} \) (with \( u, u' \in U \)), for every \( u \in U \) there exists an edge in \( H \) between a vertex of \( E_u \) and a vertex of \( E_{u'} \).

Now, let \( u \in U \) be arbitrary. Let \( e \in E_u \) and \( f \in E_U \) such that \( ef \in E(H) \). Then \( ef \in E(L(G)) \), which implies that there exist two vertices \( u' \in U \setminus \{u\} \) and \( v \in V \cap N_G(u) \) such that \( e = uv \) and \( f = uu' \). By assumption, \( V \cap N_G(u') \neq \emptyset \), say \( v' \in V \cap N_G(u') \).

Since \( uv \) and \( u'v' \) do not share a common vertex, \( (uv)(u'v') \notin E(L(G)) \). Consequently, there is no edge in \( H \) between \( uu' \) and \( u'v' \) (otherwise, it would hold that \( \text{dist}_H(uv, u'v') \leq 2 \), implying \( (uv)(u'v') \in E(H^2) = E(L(G)) \).
On the other hand, since $uv'$ and $u'v'$ are adjacent in $L(G)$, $uu'$ and $u'v'$ have a common neighbor, say $xy$, in $H$. Since $H$ is a subgraph of $H^2 = L(G)$, we may assume, without loss of generality, that $x = u'$ and denote $u'' = y$. Since all the neighbors in $H$ of vertices in $E_u'$ belong to $E_U$, we conclude that $u'' \in U \setminus \{u, u'\}$. But this gives $(uv)(u'u'') \in E(H^2)$, in contradiction to the fact that $uv$ and $u'u''$ do not share a common vertex. This completes the proof. \hfill \qedsymbol

In the next lemma, we show that the condition for a graph $G$ to be a porcupine is not only necessary for the line graph of $G$ to be a square graph but also sufficient. In addition, the line graph of every porcupine is the square of a bipartite graph.

**Lemma 4.** Let $G$ be a porcupine. Then, the line graph $L(G)$ is a square graph and has a bipartite square root.

**Proof.** Let $G$ be a porcupine and let $(U, V)$ be a partition of $V(G)$ such that $U$ is a clique and every member of $V$ is a pendant vertex in $G$. Moreover, let $x \in U$ be a vertex without neighbors in $V$.

We now construct a bipartite root $H$ of the line graph $L(G)$. By definition, $H$ has the edge set of $G$ as its vertex set. For each $u \in U \setminus \{x\}$, we put an edge between $ux$ and $uv$, for all $v \in N_G(u) \setminus \{x\}$. Formally,

\[
V(H) = V(L(G)) = E(G), \quad E(H) = \{(xu)(uv) : u \in U \setminus \{x\} \text{ and } v \in N_G(u) \setminus \{x\}\}.
\]

To illustrate this construction, Fig. 3 shows the square root of the line graph displayed in Fig. 2 constructed according to (1).

Note that if two vertices of $H$ are adjacent, then exactly one of the two corresponding edges in $G$ contains $x$. Thus, the graph $H$ is bipartite, a bipartition is given by the blocks $\{x \in E(G) : v \in V(G) \setminus \{x\}\}$ and $\{w \in E(G) : u, v \in V(G) \setminus \{x\} \text{ and } u \neq v\}$.

It remains to prove that $H^2 = L(G)$, i.e., $E(L(G)) = E(H^2)$. To see this, let $e$ and $f$ be two distinct edges of $G$. By definition, $e$ and $f$ are adjacent in $L(G)$ if and only if they share a common vertex in $G$. We will prove that the same holds in $H^2$.

We begin with the case when neither of $e$ and $f$ contains $x$. First assume that $ef \in E(L(G))$. Let $u$ be the common vertex of $e$ and $f$, say $e = uw$ and $f = uv$. In particular, $u \in U \setminus \{x\}$, and so $xu \in E(G)$. Then, $(xu)(uv), (xu)(uw) \in E(H)$ by (1). Hence, $ef \in E(H^2)$. Now we assume that $ef \in E(H^2)$. By (1), there is a vertex $u \in U$ such that $(xu)e, (xu)f \in E(H)$. But this means that $u$ is a common vertex of $e$ and $f$. Thus, $ef \in E(L(G))$.

Suppose now that $e$ and $f$ both contain $x$. Then $ef \in E(L(G))$ by definition of $L(G)$. Moreover, writing $e = ux$ and $f = vx$, the choice of $x$ implies that $u, v \in U \setminus \{x\}$. Hence, $uv$ is an edge of $G$ and a common neighbor of $e$ and $f$ in $H$. Consequently, $ef \in E(H^2)$.

Finally, we come to the case that exactly one of $e, f$ contains $x$. Without loss of generality we may assume that $x$ is contained in $e$, say $e = xu$. First assume that $ef \in E(L(G))$. Since $f$ does not contain $x$, the common vertex of $e$ and $f$ must be $u$. But then
$ef = (xu)f \in E(H)$ by (1), and so $ef \in E(H^2)$. Now we assume that $ef \in E(H^2)$. Since $H$ is bipartite, the distance from $e$ to $f$ in $H$ is odd, and so $e$ must be adjacent to $f$ in $H$. By (1), $e$ and $f$ share a common vertex. Thus, $ef \in E(L(G))$. This completes the proof.

**Lemma 5.** Let $L$ be a connected line graph with a line root $G$. If $G$ is a porcupine, then every line root of $L$ is a porcupine.

**Proof.** As Roussopoulos [20] showed, the only connected line graph the line root of which is not unique (up to isomorphism) is $K_3$, which has two line roots, the $K_3$ itself and the complete bipartite graph $K_{1,3}$. Since both $K_3$ and $K_{1,3}$ are porcupines, the statement of the lemma follows.

By combining the results just proved, we now complete the proof of Theorem 1.

**Proof of Theorem 1.** Let $L$ be a connected line graph with line root $G$. It is clear that condition 2 implies condition 1. Assume condition 1 holds, i.e., $L$ is a square graph. By Lemma 2, $G$ is a connected block graph. By Lemma 3, $G$ is a porcupine, i.e., condition 3 holds. By Lemma 5, condition 3 is well-defined in the sense that if a connected line graph has a porcupine line root, then all line roots are porcupines. Given condition 3, Lemma 4 yields condition 2. All in all, the conditions 1, 2 and 3 are equivalent.

We now argue that the above constructive results imply Corollary 1, which asserts the existence of a linear time algorithm for computing, given a line graph $L$, a root of $L$ (if one exists).

**Proof of Corollary 1.** Without loss of generality, we can assume that $L$ is connected. Theorem 1 says that $L$ is a square graph if and only if the line root of $L$ is a porcupine. Using the algorithm of Roussopoulos [20], we compute in linear time a line root $G$ of $L$. Whether $G$ is a porcupine can be checked in linear time (even in the size of $G$) by a straightforward algorithm.

In Lemma 4 it is shown how a bipartite square root of $L$ is constructed. This construction can be done in linear time, again by a straightforward algorithm.

In conclusion, we give a proof of Theorem 2, giving a characterization of the line graphs of porcupines.

**Proof of Theorem 2.** Let $L$ be a graph.

First we assume that $L$ has a line root $G$ which is a porcupine. We have to find a set of cliques of $L$ with the properties described in the lemma. Recall that, since $G$ is a porcupine, there exists a partition of $V(G)$ into two sets $U$ and $V$ with the following properties. The set $U$ is a non-empty clique, every vertex of $V$ has exactly one neighbor, and this neighbor is contained in $U$. Moreover, there is at least one vertex of $U$ that does not have a neighbor in $V$.

Let $n = |U|$ and $U = \{u_1, \ldots, u_n\}$. For all $1 \leq i \leq n$, we set $C_i := \{e \in E(G) : e$ is incident to $u_i$ in $G\}$. Obviously, every $C_i$ is a clique in $L$. 11
To see (a), note that every vertex of $L$ is an element of at least one $C_i$, since every edge of $G$ is incident to at least one vertex of $U$. Moreover, the cliques $\{C_i : 1 \leq i \leq n\}$ cover all edges, as is seen as follows. If two edges $e_1$ and $e_2$ are adjacent in $G$, they share a vertex, say $u_j$, of $U$, by the choice of $U$ and $V$. Hence, $e_1$ and $e_2$ both are members of the clique $C_j$. Thus, the cliques $\{C_i : 1 \leq i \leq n\}$ cover all vertices and edges of $L$.

Since $G$ is a porcupine, there exists at least one vertex $u_i \in U$ that has no neighbors in $V$, and therefore has exactly $n-1$ incident edges in $G$, namely the edges connecting $u_i$ and $u_j$, $1 \leq j \leq n$, $i \neq j$. Thus, the clique $C_i$ contains exactly $n-1$ vertices and so (b) holds.

To see (c), assume $e \in C_i \cap C_j$ where $i \neq j$. The edge $e$ of $G$ has to be to $u_i$ and $u_j$. Hence, $e$ can not be contained in any other clique.

For (d), note that, since $U$ is a clique in $G$, for every pair $C_i$ and $C_j$ of distinct cliques of the set, there exists an edge $e \in C_i \cap C_j$, namely $e = u_iu_j$. It is clear that there is exactly one such edge.

Now let $n \geq 1$ and let $L$ have a collection of cliques $\{C_i : 1 \leq i \leq n\}$ with the properties described in the lemma. We have to show that there is a line root $G$ of $L$ which a porcupine. For this, we rename the vertices of $L$ as follows. For every pair $i$ and $j$, where $1 \leq i < j \leq n$, we denote the unique vertex contained in $C_i \cap C_j$ by $x_{i,j}$. Due to (d), this is well-defined. For every $i$, let $k_i$ be the number of vertices contained in $C_i$ only, and let us enumerate these vertices as $x_{1,i}^1, x_{1,i}^2, \ldots, x_{1,i}^{k_i}$. By (a), all vertices of $L$ are now uniquely renamed.

In the following, we construct a line root $G$ of $L$ for which we prove that it is a porcupine.

We start with $G$ being the empty graph. We insert a vertex $u_i$ into $G$ for every $1 \leq i \leq n$. For every $1 \leq i \leq n$ and every $1 \leq j \leq k_i$, we add a vertex $v_i^j$ and the edge $e_i^j = u_iv_i^j$ to $G$. We associate this edge $e_i^j$ of $G$ with the vertex $x_i^j$ of $L$. For every vertex $x_{k,l}$, where $1 \leq k < l \leq n$, we add the edge $u_ku_l$ to $G$. We associate this edge of $G$ with the vertex $x_{k,l}$ of $L$. This completes the construction of $G$.

For every vertex contained in a clique of the set $\{C_i : 1 \leq i \leq n\}$, we added exactly one edge to $G$. By (a), we have $\bigcup_{i=1}^n C_i = V(L)$. Hence, the vertices of $L$ are in a one-to-one correspondence with the edges of $G$. We will now show that this bijection gives an isomorphism between $L$ and $L(G)$.

For this, let $x, y$ be two adjacent vertices of $L$. Then $x, y \in C_i$ for some $1 \leq i \leq n$, by (a). It follows from our construction that the edges associated to $x$ and $y$ in $G$ share the vertex $u_i$, and are therefore adjacent as vertices of $L(G)$. Now let $e, f$ be two distinct incident edges in $G$. Moreover, let $x_e$ (resp. $x_f$) be the vertex of $L$ associated with $e$ (resp. $f$). By construction, $e$ and $f$ have to share some vertex $u_i$ in $G$. Hence, $x_e, x_f \in C_i$ and so $x_e x_f \in E(L)$. Thus, $L(G)$ is isomorphic to $L$.

It remains to show that $G$ is a porcupine. By (d) and our construction, the set $U := \{u_i : 1 \leq i \leq n\}$ is a clique. Moreover, every member of the set $V := \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq k_i\}$ has exactly one neighbor, and this neighbor is contained in $U$. Let, without loss of generality, $C_n$ be the clique in $\{C_i : 1 \leq i \leq n\}$ of cardinality $n-1$ (such a clique exists by (b)). The clique $C_n$ has to share a common vertex with the clique $C_i$, for all $1 \leq i \leq n-1$. By (c), every vertex of $C_n$ belongs to exactly one other clique $C_j$. 12
Therefore, \( N_G(u_n) = \{u_1, u_2, \ldots, u_{n-1}\} \) by construction. Thus \( G \) is a porcupine and this completes the proof.

5 Concluding remarks

In view of the characterization of line graphs that are square graphs, given by Theorem 1, a natural next question would be to characterize line graphs that are \( k \)-th powers, for \( k \geq 3 \).

Using the fact that line graphs are \( K_5^- \)-free (cf. Proposition 1), it is easy to show that for \( k \geq 3 \), the only connected line graphs that are \( k \)-th powers of graphs are the complete graphs. For the sake of completeness, we state this observation and its simple proof below.

**Proposition 3.** Let \( L \) be a connected line graph and let \( k \geq 3 \). Then, graph \( L \) is the \( k \)-th power of some graph if and only if \( L \) is a complete graph.

**Proof.** Let \( L \) be a connected line graph and let \( k \geq 3 \). It is clear that if \( L \) is a complete graph, it is the \( k \)-th power of itself. On the other hand, let \( L \) be the \( k \)-th power of some graph \( H \). Suppose that \( L \) is not a complete graph. Then, the diameter of \( H \) is \( k + 1 \) or greater. Let \( x, y \) be two vertices of \( H \) of distance \( k + 1 \). Let \( P = (x = v_0, v_1, \ldots, v_k, v_{k+1} = y) \) be a shortest path from \( x \) to \( y \). In \( L \), both sets \( \{x, v_1, v_2, v_3\} \) and \( \{v_1, v_2, v_3, y\} \) form a clique, and \( x \) is not adjacent to \( y \). Hence, \( L[\{x, v_1, v_2, v_3, y\}] \) is isomorphic to \( K_{5^-} \). This is a contradiction to Proposition 1.

Theorem 1 and its proof showed that line graphs that have a square root can be recognized in linear time. However, to argue only about polynomial time solvability of recognizing line graphs that are squares, one does not need the full characterization of such graphs. It suffices to use the part of Theorem 1 stating that line graphs with a square root have a bipartite square root, together with a polynomial time algorithm to verify whether a given graph admits a bipartite square root [11]. More generally, for a graph property \( \pi \), let us say that a graph class \( C \) is \( \pi \)-square-rooted (or just \( \pi \)-rooted, for short) if every graph in \( C \) with a square root has a square root with property \( \pi \). In this terminology, Theorem 1 implies that line graphs are bipartite-rooted, and [18, Corollary 3] implies that every connected trivially perfect graph is split-rooted. (For definitions of trivially perfect and split graphs, see, e.g., [5].)

Clearly, if there is a polynomial time algorithm to verify whether a given graph admits a square root with property \( \pi \), then the square graph recognition problem is polynomially solvable within every \( \pi \)-rooted graph class. For example, due to the results in [8], [11], [12], [14], and [15], respectively, property \( \pi \) with this nice feature can be any of the following: of girth at least six, bipartite, proper interval, block, strongly chordal split. We believe that a more systematic study of \( \pi \)-rooted graph classes deserves further attention. Besides having a potential for revealing new interesting relations among well known graph classes, it might help to identify further polynomially solvable cases of the square graph recognition problem.
References


