Interval total colorings of bipartite graphs

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Key words: total coloring, interval coloring, bipartite graph, tree, complete bipartite graph

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1 Introduction

A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. The concept of total coloring was introduced by V. Vizing [15] and independently by M. Behzad [3]. The total chromatic number $\chi''(G)$ is the smallest number of colors needed for total coloring of G. In 1965 V. Vizing and M. Behzad conjectured that $\chi''(G) \leq \Delta(G) + 2$ for every graph G [3,15], where $\Delta(G)$ is the maximum degree of a vertex in G. This conjecture became known as Total Coloring Conjecture [5]. It is known that Total Coloring Conjecture holds for cycles, for complete graphs, for bipartite graphs, for complete multipartite graphs [17], for graphs with a small maximum degree [6,11,14], for graphs with minimum degree at least $\frac{3}{4}|V(G)|$ [5] and for planar graphs G with $\Delta(G) \neq 6$ [5,7,13,16]. M. Rosenfeld [11] and N. Vijayaditya [14] independently proved that the total chromatic number of graphs G with $\Delta(G) = 3$ is at most 5. A. Kostochka in [6] proved that the total chromatic number of graphs with $\Delta(G) \leq 5$ is at most 7. The general upper bound for the total chromatic number was obtained by M. Molloy and B. Reed [8], who proved that $\chi''(G) \leq \Delta(G) + 10^{26}$ for every graph G. The exact value of the total chromatic number is known only for paths, cycles, complete and complete bipartite graphs, n-dimensional cubes, complete multipartite graphs of odd order [5], outerplanar graphs [18] and planar graphs G with $\Delta(G) \geq 9$ [4,5,7,16].

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The key concept discussed in this work is the following. Given a graph G, we say that G is interval total colorable if there is $t \geq 1$ for which G has a total coloring with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by i, $i = 1, 2, \ldots, t$, and the edges incident with each vertex v together with v are colored by $d_G(v) + 1$ consecutive colors, where $d_G(v)$ is the degree of the vertex v in G.

The concept of interval total coloring [9,10] is a new one in graph coloring, synthesizing interval colorings [1,2] and total colorings. The introduced concept is valuable as it connects to the problems of constructing a timetable without a "gap"and it extends to total colorings of graphs one of the most important notions of classical mathematics - the one of continuity.

In this work interval total colorings of bipartite graphs are investigated.

2 Main results

All graphs considered in this work are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. An (a, b)-biregular bipartite graph G is a bipartite graph G with the vertices in one part have degree a and the vertices in the other part have degree b. An interval total t-coloring of a graph G is a total coloring of G with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by i, $i = 1, 2, \ldots, t$, and the edges incident to each vertex v together with v are colored by $d_G(v) + 1$ consecutive colors. The set of all interval total colorable graphs is denoted by $\mathfrak{T}$. For a graph $G \in \mathfrak{T}$, the least (the minimum span) and the greatest (the maximum span) values of t for which G has an interval total t-coloring is denoted by $w_\tau(G)$ and $W_\tau(G)$, respectively. Clearly,

$$\chi''(G) \leq w_\tau(G) \leq W_\tau(G) \leq |V(G)| + |E(G)| \text{ for every graph } G \in \mathfrak{T}.$$

First, we give a general upper bound for the maximum span in interval total colorings of bipartite graphs.

Theorem 1. If G is a connected bipartite graph and $G \in \mathfrak{T}$, then

$$W_\tau(G) \leq 2|V(G)| - 1.$$

Note that the bound of Theorem 1 is sharp for the simple path P_n, since $W_\tau(P_n) = 2n - 1$ for any $n \in \mathbf{N}$.

Next, we show that many bipartite graphs such as subcubic bipartite graphs, regular bipartite graphs, trees, complete bipartite graphs, n-dimensional cubes, $(2, b)$-biregular bipartite graphs, doubly convex bipartite graphs, grids and some classes of bipartite graphs with maximum degree 4 have interval total colorings. Moreover, we also obtain some bounds for the minimum span and the maximum span in interval total colorings of these graphs. In particular, we prove the following theorems.

Theorem 2. If G is a bipartite graph with $\Delta(G) \leq 3$, then $G \in \mathfrak{T}$ and $w_\tau(G) \leq 5$.

Theorem 3. If G is an r-regular bipartite graph with $r \geq 2$, then $G \in \mathfrak{T}$ and $r+1 \leq w_\tau(G) \leq r+2$.

Note that for any $r \geq 3$, there is an r-regular bipartite graph such that $G \in \mathfrak{T}$ and $w_\tau(G) = r+2$. In [12], it was proved that the problem of determining whether $\chi''(G) = 4$ is NP-complete even for a cubic bipartite graph G. Therefore we can conclude that verification whether $w_\tau(G) = r+1$ for an r-regular bipartite graph G with $r \geq 3$ is also NP-complete.

Theorem 4. If T is a tree, then $T \in \mathfrak{T}$ and $w_\tau(T) \leq \Delta(T) + 2$.

Theorem 5. If $m+n+2-\gcd(m,n) \leq t \leq m+n+1$, where $\gcd(m,n)$ is the greatest common divisor of m and n, then the complete bipartite graph $K_{m,n}$ has an interval total t-coloring for any $m,n \in \mathbf{N}$.

Theorem 6. For any $m,n \in \mathbf{N}$

$$W_\tau(K_{m,n}) = \begin{cases} m+n+1, & \text{if } m=n=1, \\ m+n+2, & \text{otherwise.} \end{cases}$$

Theorem 7. For the n-dimensional cube Q_n, we have $Q_n \in \mathfrak{T}$ with

$$w_\tau(Q_n) = \chi''(Q_n) \text{ and } W_\tau(Q_n) \geq \tfrac{(n+1)(n+2)}{2} \text{ for any } n \in \mathbf{N}.$$

Moreover, for the n-dimensional cube Q_n, we show that if $w_\tau(Q_n) \leq t \leq \frac{(n+1)(n+2)}{2}$, then Q_n admits an interval total t-coloring.

Theorem 8. If G is a $(2,b)$-biregular bipartite graph with $b \geq 3$, then $G \in \mathfrak{T}$ and $b+1 \leq w_\tau(G) \leq b+2$.

Finally, we show that there are bipartite graphs which have no interval total coloring. The smallest known bipartite graph with 26 vertices and maximum degree 18 that is not interval total colorable was obtained by A. Shashikyan.

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