# Lattice Polyhedra and Submodular Flows

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Lattice polyhedra were introduced by Hoffman and Schwartz as a common framework for various discrete optimization problems. They are specified by a ternary matrix whose row set forms a *consecutive, supermodular* lattice and some submodular rank function (the terms "sub"-and "supermodular" can also be interchanged). Though lattice polyhedra are known to be integral, so far no combinatorial algorithm could have been found for the corresponding linear optimization problem. We show that the important class of distributive lattice polyhedra in which the underlying lattice is both, sub-and supermodular can be reduced to Edmonds-Giles polyhedra. Thus, submodular flow algorithms can be applied to this class of lattice polyhedra.

## 1 Introduction

A large class of discrete optimization problems allow a formulation as integer linear program with underlying ternary matrix: given a matrix $A \in \{-1,0,1\}^{L\times E}$, some weight function $w \in \mathbb{R}^E$, lower and upper bounds $c, d \in \mathbb{R}^E$ and some "rank" function $f \in \mathbb{R}^L$ find an integral solution of

$$(LP) \quad \max_{x\in\mathbb{R}^E}\{w^T x \mid Ax \le f,\ c \le x \le d\}.$$

This problem is easily seen to be $\mathcal{NP}$-hard even if restricted to binary matrices. Therefore, we are looking for more special structures of the polyhedron

$$\mathbb{P}(A, f) = \{x \in \mathbb{R}^E \mid Ax \le f,\ c \le x \le d\}.$$

A promising class is that of *lattice polyhedra* which were introduced by Hoffman and Schwartz [HS78] and shown to be integral. The name comes from a certain, very general, lattice structure on $A$ on which $f$ is submodular.

**Definition 1 (Lattice polyhedra)** *Let $A \in \{-1,0,1\}^{L\times E}$ be a matrix with entries $\chi(i,e)$ for $i \in L$ and $e \in E$, and let $c, d \in \mathbb{R}^E$ and $f \in \mathbb{R}^L$. Then the polyhedron*

$$\mathbb{P}(A, f) = \{x \in \mathbb{R}^E \mid Ax \le f,\ c \le x \le d\}$$

*is called a* lattice polyhedron *if the row index set $L$ forms a lattice $L = (L, \preceq, \wedge, \vee)$ on which $f$ is submodular, i.e., $f$ satisfies*

$$f(i) + f(j) \geq f(i \wedge j) + f(i \vee j) \quad \forall i, j \in L,$$

*and where for all $i, j, k \in L$ and all $e \in E$ the following three hold:*

(C1) *if $i \prec j \prec k$ and $\chi(i, e) = \chi(k, e) = t \neq 0$, then $\chi(j, e) = t$,*

(C2) *if $i \prec j$, then $\chi(i, e) \cdot \chi(j, e) \geq 0$, and*

(C3) $\chi(i, e) + \chi(j, e) \leq \chi(i \vee j, e) + \chi(i \wedge j, e)$.

Analogously, if, in the above definition, function $f$ is supermodular and (C3) is replaced by

(C3′) $\chi(i, e) + \chi(j, e) \geq \chi(i \vee j, e) + \chi(i \wedge j, e)$,

the polyhedron

$$\mathbb{P}'(A, f) = \{x \in \mathbb{R}^E \mid Ax \geq f,\ c \leq x \leq d\}$$

is also called a lattice polyhedron. The lattice $L$ is called *consecutive* if properties (C1) and (C2) are satisfied. If $L$ satisfies (C3) (or (C3′)), we call it *supermodular* (or *submodular*).

Lattice polyhedra form a common framework for various combinatorial structures such as polymatroids, the intersection of polymatroids, and Edmonds-Giles polyhedra. Several min-max results for combinatorial structures can be derived from the following theorem:

**Theorem 1 ([HS78], [H82])** *If $f, c$ and $d$ are integral, then all vertices of lattice polyhedra are integral.*

However, this integrality result is only a structural existence theorem without algorithmic foundation. While several greedy-type algorithms have been developed for special instances of lattice polyhedra in the last decades (see e.g. [**?**], [FP08], [FK96], [FK00], [2], [E70], [3], [DH03], up to now no combinatorial algorithm could have been found for lattice polyhedra in its general form, even if the polyhedra are restricted to binary matrices.

A very important class of lattice polyhedra is that of *distributive lattice polyhedra*, in which the lattice $(L, \preceq, \wedge, \vee)$ is distributive and (C3) is satisfied with equality.

Let us first recall some basic facts about distributive lattices: A lattice $(L, \preceq, \wedge, \vee)$ is called *distributive* if the binary operators $\wedge, \vee$ satisfy the distributive laws. Alternatively, distributive lattices can be characterized by the exclusion of the sublattices $N_5$ and $M_3$. By a theorem of Birkhoff, a distributive lattice $L$ is isomorphic to the lattice $\mathcal{D}(P)$ of all ideals [1] of poset $(P, \preceq)$ on the set $P$ of join-irreducible elements [2] of $L$. (For more details about lattices the reader is referred to [B91].)

Beside classical examples of combinatorial structures such as polymatroids, the intersection of polymatroids, or submodular systems, distributive lattice polyhedra also cover Edmonds-Giles polyhedra (see below). Furthermore, we show in Theorem 2 below that a large class of lattice polyhedra is in fact distributive. Finally, we show in Theorem 3 that distributive lattice polyhedra can in fact be reduced to Edmonds-Giles polyhedra for which several efficient algorithms exist.

**Theorem 2** *Let $\mathbb{P}(A, f)$ be a lattice polyhedron in which any two rows of $A$ are distinct and property (C3) is satisfied with equality. Then the underlying lattice $(L, \preceq, \wedge, \vee)$ is distributive.*

**Proof:** For the sake of contradiction, assume that $L$ is not distributive, i.e., that it contains an $N_5$- or an $M_3$-sublattice. Then there exist five distinct elements $i, j, k, l, m \in L$ such that

$$l = i \wedge j = i \wedge k \quad \text{and} \quad m = i \vee j = i \vee k.$$

Since $\chi(j) \neq \chi(k)$ by the assumption, choose some element $e \in E$ with $\chi(j, e) \neq \chi(k, e)$. Since (C3) is satisfied with equality, it follows that

$$\chi(l, e) + \chi(m, e) = \chi(i, e) + \chi(j, e) = \chi(i, e) + \chi(k, e),$$

which implies $\chi(j, e) = \chi(k, e)$, a contradiction. $\square$

**Edmonds-Giles polyhedra.** Let $G = (V, E)$ be a connected directed graph and $\mathcal{F} \subseteq 2^V$ be a ring family of subsets of vertex set $V$, (i.e., $\mathcal{F}$ is union-and intersection-closed). Given a submodular function $f : \mathcal{F} \to \mathbb{R}$ and lower and upper bounds on the edges $c, d : E \to \mathbb{R}$, the Edmonds-Giles polyhedron is

$$\mathbb{P}(G, \mathcal{F}, f) = \{x \in \mathbb{R}^E \mid x(\Delta^+(S)) - x(\Delta^-(S)) \leq r(S) \ \forall S \in \mathcal{F}, c \leq x \leq d\},$$

where $\Delta^+(S)$ and $\Delta^-(S)$ denote, respectively, the sets of arcs leaving $S$ and of entering $S$. (In the original definition, $\mathcal{F}$ is a crossing family on which $f$ is crossing submodular. However, it suffices to consider the case of ring families with submodular $f$, as the more general crossing case can be reduced to it using the Dilworth truncation (see e.g. [Fuj91]).) Edmonds and Giles [EG77] proved that $\mathbb{P}(G, \mathcal{F}, f)$ is integral, and several algorithms for the corresponding linear optimization problem, called the *submodular flow problem*, have been established (see e.g., the survey paper [FI00]. Almost all submodular flow algorithms are based on generalizations of different min-cost-flow algorithms).

Also Edmonds-Giles polyhedra turn out to be distributive lattice polyhedra: given an Edmonds-Giles polyhedron $\mathbb{P}(G, \mathcal{F}, f)$ consider the collection of ordered pairs

$$L = \{(\Delta^+(S), \Delta^-(S)) \mid S \in \mathcal{F}\} \subseteq 3^E$$

[1] A subset $I \subseteq P$ is an *ideal* of poset $(P, \preceq)$ if $i \preceq j$ and $j \in P$ implies $i \in P$

[2] An element $i \in L$ is *join-irreducible* if $i = j \vee k$ implies $i = j$ or $i = k$

partially ordered by

$$(\Delta^+(S), \Delta^-(S)) \preceq (\Delta^+(T), \Delta^-(T)) \iff S \subseteq T$$

and with join- and meet-operations

$$\begin{aligned}(\Delta^+(S), \Delta^-(S)) \vee (\Delta^+(S), \Delta^-(S)) &= (\Delta^+(S \cup T), \Delta^-(S \cup T))\\ (\Delta^+(S), \Delta^-(S)) \wedge (\Delta^+(T), \Delta^-(T)) &= (\Delta^+(S \cap T), \Delta^-(S \cap T)).\end{aligned}$$

Also for all $S \subseteq V$ and $e \in E$ define

$$\chi(S,e) = \begin{cases} 1 & e \in \Delta^+(S) \\ -1 & e \in \Delta^-(S) \\ 0 & \text{otherwise} \end{cases}$$

Then $(L, \preceq, \wedge, \vee)$ with such a $\chi$ is a consecutive, sub- and supermodular lattice.

While it seems that the Edmonds-Giles polyhedra form a special class of distributive lattice polyhedra, we will see that they are in fact equivalent, i.e., we show that any distributive lattice polyhedron can be reduced to some Edmonds-Giles polyhedron. For this, we construct an auxiliary digraph $G$ whose vertices correspond to the join-irreducible elements $P$ of $L$ and whose edges correspond to the elements in $E$. We then show that the lattice polyhedron is equivalent to the Edmonds-Giles polyhedron $\mathbb{P}(G, \mathcal{D}(P), f)$, i.e., we show (in the appendix)

**Theorem 3** *If $\mathbb{P}(A, f)$ is a distributive lattice polyhedron, then there exists an auxiliary digraph $G = (P, E)$ whose vertices correspond to the join-irreducible elements of $L$ such that*

$$\begin{aligned}\mathbb{P}(A, f) &= \{x \in \mathbb{R}^{E'} \mid \forall I \in \mathcal{D}(P) : x(\Delta^+(I)) - x(\Delta^-(I)) \leq f(I),\ c \leq x \leq d\}\\ &= \mathbb{P}(G, \mathcal{D}(P), f)\end{aligned}$$

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