On Reed's Conjecture in Triangle-Free Graphs

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1 Introduction

The problem of finding the chromatic number of a graph G, i.e. the minimum number of colors needed to assign distinct colors to adjacent nodes of G, is NP-complete. However, there are some known bounds including the trivial lower bound

$$\chi(G) \geq \omega(G)$$

and the upper bound provided by Brooks [2]

$$\chi(G) \leq \Delta(G) + 1,$$

where $\chi(G)$ denotes the chromatic number, $\Delta(G)$ the maximum degree and $\omega(G)$ the number of nodes of a largest clique in G.

In [6], Reed conjectured that

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil .$$

Note that this upper bound rounds up the arithmetic average of the previous mentioned upper and lower bounds.

In the following all graphs are simple, undirected and connected. $V(G)$ and $E(G)$ denote the nodes and edges of G respectively. The degree of $v \in V(G)$ ($deg(v)$) gives the number of neighbors of v in G.

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Reed [6] proved the existence of a constant Δ_0 so that the conjecture holds for graphs with $\Delta(G) \geq \Delta_0$ and $\omega(G) \geq \lfloor (1 - \frac{1}{70000000})\Delta(G) \rfloor$.

Other graph classes for which the validity of Reed's conjecture could be shown have been found, among them the classes listed in Proposition 1. This list consists of graph classes for which the conjecture follows straightforward from their definition or for which other authors (sources given in parantheses) have shown the conjecture holds.

Proposition 1 *For the following graphs the conjecture holds:*

- *complete graphs*
- *perfect graphs*
- *circles*
- *graphs with size of maximum independent set $\alpha(G) = 2$ ([5])*
- *line graphs ([4])*
- *triangle-free graphs with minimum degree $\delta(G) \geq \frac{n}{3}$ ([1])*
- *triangle-free graphs with chromatic number $\chi(G) \leq 4$ (consequence of Brooks' theorem)*

In this work we focus on triangle-free graphs ($\omega(G) = 2$), for which Reed's conjecture reads as follows:

$$\chi(G) - 1 \leq \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil .$$

In order to investigate the conjecture we search for a triangle-free counterexample with a minimum number of nodes.

Gathering properties of a minimal order counterexample, we can continue a list already 'started' in [3]. Some of these properties turn out to be quite fruitful, in particular the fact that a minimal counterexample has to be vertex-color-critical, i.e. $\chi(G - v) < \chi(G)$ for any vertex v of G.

Definition 2 (heavy edge) *Let G be a graph. Then we call $(x, y) \in E(G)$ a heavy edge if $deg(x) = deg(y) = \Delta(G)$.*

Definition 3 (heavy circle) *Let G be a graph and $C \subseteq V(G)$ a circle of G. We call C a heavy circle if*

$$v \in V(C) \Rightarrow deg(v) \geq \Delta(G) - 2.$$

2 Results

Theorem 4 *Let G be a triangle-free minimal counterexample to Reed's conjecture. Then $\Delta(G) = 2\chi(G) - 5$.*

Theorem 5 *Let G be a triangle-free minimal counterexample to Reed's conjecture. Then there exists at least one heavy edge.*

Theorem 6 *Let G be a triangle-free minimal counterexample to Reed's conjecture. Then there exists at least one heavy circle C of odd length.*

Moreover, we describe a construction which embeds an arbitrary triangle-free graph G in a regular triangle-free graph preserving $\chi(G)$ and $\Delta(G)$. From this we get that it suffices to prove Reed's conjecture for regular triangle-free graphs.

Theorem 7 *If Reed's conjecture holds for all regular triangle-free graphs, it holds for all triangle-free graphs.*

References

[1] S. Brandt, S. Thomassé (2005): *Dense triangle-free graphs are four-colorable: A solution to the Erdös-Simonovits problem.* To appear in Journal of Combinatorial Theory B.
http://www.lirmm.fr/ thomasse/liste/vega11.pdf

[2] R.L. Brooks (1941): *On colouring the nodes of a network.* Proceedings of the Cambridge Philosophical Society 37, 194 - 197.

[3] D. Gernert, L. Rabern (2007): *A knowledge-based system for graph theory, demonstrated by partial proofs for graph coloring problems.* MATCH Commun. Math. Comput. Chem., Vol. 58, 445 - 460.

[4] A.D. King, B.A. Reed, A. Vetta (2007): *An upper bound for the chromatic number of line Graphs.* European Journal Of Combinatorics, Vol. 28(8), 2182-2187.

[5] M. Molloy (1991): *Chromatic neighbourhood sets.* Journal Of Graph Theory, Vol. 31, 303 - 311.

[6] B. Reed (1998): *ω,Δ and χ (OMEGA, DELTA AND CHI).* Journal Of Graph Theory, Vol. 27, 177 - 212.