Radio Labeling Cartesian Graph Products

Cynthia Wyels [a]

[a] *California State University Channel Islands*

Maggy Tomova [b]

[b] *University of Iowa*

Key words: radio number, radio labeling, Cartesian product

1 Introduction

Radio labeling is derived from the assignment of radio frequencies (channels) to a set of transmitters. The frequencies assigned depend on the geographical distance between the transmitters: the closer two transmitters are, the greater the potential for interference between their signals. Thus when the distance between two transmitters is small, the difference in the frequencies assigned must be relatively large, whereas two transmitters at a large distance may be assigned frequencies with a small difference.

The use of graphs to model the "channel assignment" problem was first proposed by Hale in 1980 [5]. Several schemes for distance labeling were subsequently introduced and have been extensively studied; Chartrand et al introduced the variation known as radio labeling in 2001 [2].

In the graph model of the channel assignment problem, the vertices correspond to the transmitters, and graph distance plays the role of geographical distance. We assume all graphs are connected and simple. The *distance* between two vertices u and v of a graph G, $d(u, v)$, is the length of a shortest path between u and v. The *diameter* of G, $diam(G)$, is the maximum distance, taken over all pairs of vertices of G. A *radio labeling* of a graph G is then defined to be a function $c : V(G) \to \mathbb{Z}_+$ satisfying

$$d(u, v) + |c(u) - c(v)| \geq 1 + diam(G) \tag{1}$$

for all distinct pairs of vertices $u, v \in V(G)$. The *span* of a radio labeling c is the maximum integer assigned by c. The *radio number* of a graph G, $rn(G)$, is the minumum span, taken over all radio labelings of G [1].

As Liu and Zhu write, “It is surprising that determining the radio number seems a difficult problem even for some basic families of graphs.” [9] The radio number is known exactly for only a few graph families, including paths and cycles [9] and the squares of paths [8] and cycles [7]; wheels and gears [3], some generalized prisms [11], and Cartesian products of a cycle with itself [10]. Meanwhile, bounds for the radio numbers of trees [6], ladders [4], and square grids [1] have been identified, while the radio number of cubes of the cycles C_n^3 for $n \leq 20$ and $n \equiv 0$, 2, or 4 $\pmod 6$ is known [12].

In this investigation we focus as follows:

Question: *What may be said about the radio number of the Cartesian product of two graphs?*

The Cartesian product of two graphs G and H has vertex set $V(G\square H) = V(G) \times V(H) = \{(g,h) \mid g \in V(G) \text{ and } h \in V(H)\}$. The edges of $G\square H$ consist of those pairs of vertices $\{(g,h), (g',h')\}$ satisfying $g = g'$ and h is adjacent to h' in H or $h = h'$ and g is adjacent to g' in G. We note the following facts about Cartesian products:

- The order of a product is the product of the orders of the factor graphs, i.e., $G\square H$ has $|V(G)| \cdot |V(H)|$ vertices.
- Distances in products are sums of distances between corresponding vertices in factor graphs, i.e., $d_{G\square H}\left((g_1,h_1),(g_2,h_2)\right) = d_G(g_1,g_2) + d_H(h_1,h_2)$.
- In particular, the diameter of a product is the sum of the diameters of the factors, i.e., $diam(G\square H) = diam(G) + diam(H)$.

In Section 2 we provide three lower bounds for the radio number of a Cartesian product, each of which outperforms the others in specific cases. Two upper bounds are provided in Section 3, along with some comments as to their efficacy.

2 Lower Bounds

Our first bound follows directly from the fact that a radio labeling is an injection. As such, the span of any radio labeling may never be less than the number of vertices of the associated graph.

[1] We use the convention, established in [2], that the co-domain of a radio labeling is $\mathbb{Z}_+ = \{1, 2, \ldots\}$. Some authors use $\{0, 1, 2, \ldots\}$ as the co-domain; radio numbers specified using the non-negative integers as co-domain are one less than those determined using the positive integers.

Theorem 1 *(Vertex Lower Bound)*

$$rn(G\Box H) \geq |V(G)| \cdot |V(H)|.$$

This lower bound is tight for the product of the Petersen graph with itself.

The second lower bound is stated in terms of the radio numbers of the factor graphs.

Theorem 2 *(Radio Number Lower Bound)Let G and H be graphs.*

$$rn(G\Box H) \geq rn(G) + rn(H) - 1.$$

This bound outperforms the Vertex Lower Bound on prism graphs, which are products of 2-paths with n-cycles. These prism graphs have $2n$ vertices (so the Vertex Lower Bound states the radio number is not less than $2n$); the Radio Lower Bound gives a lower bound that is $O(n^2)$.

To specify the third lower bound, an additional term, the "gap," must be introduced. Essentially, the gap of G is the smallest possible difference between the ith and $(i+2)$nd largest labels in a radio labeling of G.

Definition 3 *Let c be a radio labeling of a graph G, and let $\{x_1, x_2, \ldots, x_n\}$ be the vertices of G, arranged so that $c(x_i) < c(x_j)$ whenever $i < j$. Define $\phi(G,c)$ to be the smallest integer satisfying $\phi(G,c) \geq c(x_{i+2}) - c(x_i)$ for all $i \in \{1, 2, \ldots, n-2\}$. Finally, define $\phi(G)$ to be the minimum of $\phi(G,c)$ taken over all radio labelings c of G.*

Our third lower bound is expressed in terms of this gap.

Theorem 4 *(Gap Lower Bound)*

$$rn(G\Box H) \geq \left(\left\lfloor \frac{1}{2}|V(G)| \cdot |V(H)| \right\rfloor - 1\right)(\phi(G) + \phi(H) - 2) + a,$$

where $a = 1$ when $|V(G)| \cdot |V(H)|$ is odd and $a = 2$ otherwise.

This lower bound is again sharp for the product of the Petersen graph with itself. Moreover, it is significantly more effective than the Vertex and Radio Lower Bounds on products of cycles with themselves and products of paths with themselves. The Vertex and Radio Lower Bounds give lower bounds that are $O(n^2)$ for each; the Gap Lower Bound gives $O(n^3)$ bounds. (We note that the radio number of both families of products has been shown to be $O(n^3)$ [1], [10].)

3 Upper Bounds

In this section we turn our attention to determining upper bounds for radio numbers of Cartesian products of graphs. The radio condition (1) depends on distance and diameter; without knowledge of specific distances between pairs of vertices in the product graph, it is unreasonable to expect upper bounds to be sharp. Nonetheless, we present two upper bounds with varied hypotheses.

Theorem 5 *Let G be a graph with diameter 2 and $rn(G) = |V(G)| = n$. Then*

$$rn(G\Box G) \leq n^2 + 2(2n-2) + \left(2\left\lfloor\frac{n}{2}\right\rfloor - 1\right)\left\lfloor\frac{n-1}{2}\right\rfloor.$$

The statement of the next theorem is similar in spirit.

Theorem 6 *Assume G and H are graphs satisfying $rn(G) = |V(G)| = n$, $rn(H) = |V(H)| = m$, and $diam(G) - diam(H) \geq 2$. Then*

$$rn(G\Box H) \leq diam(G)(n+m-2) + 2mn - 2m - 4n + a,$$

where $a = 7$ when m and n have opposite parity, $a = 8$ when both m and n are odd, and $a = 6$ when both m and n are even.

To establish both of these theorems, one must replace the radio condition (1) with conditions sufficient for labelings of the products to be radio labelings. As nothing is assumed regarding distances between vertices, the bounds necessarily involve products of the diameters of the factor graphs. It would be of interest to investigate additional upper bounds resulting from removing the hypotheses that the factor graphs have the smallest possible radio numbers, or by including additional hypotheses regarding structure of the factor graphs.

References

[1] Calles, L., Gomez, H., Tomova, M., and Wyels, C., Bounds for the radio number of square grids, in preparation.

[2] Chartrand, G., Erwin, D., Zhang, P. and Harary, F., Radio labelings of graphs, Bull. Inst. Combin. Appl., **33** (2001), 77–85.

[3] Fernandez, C., Flores, A., Tomova, M., and Wyels, C., The radio number of gear graphs, preprint available at front.math.ucdavis.edu/0809.2623.

[4] Flores, J., Lewis, K., Tomova, M., and Wyels, C., The radio number of ladder graphs, in preparation.

[5] Hale, W.K.,Frequency assignment: theory and application, Proc. IEEE, **68** (1980), 1497–1514.

[6] Liu, D.D.-F., Radio number for trees, Discrete Math. **308** (2008), no. 7, 1153–1164.

[7] Liu, D.D.-F., and Xie, M., Radio number for square cycles, Congr. Numer. **169** (2004), 105125.

[8] Liu, D.D.-F., and Xie, M., Radio number for square paths, preprint available at www.calstatela.edu/faculty/dliu/ArsComFinal.pdf.

[9] Liu, D.D.-F., Zhu, X., Multilevel distance labelings for paths and cycles, SIAM J. Discrete Math. **19** (2005), No. 3, 610–621.

[10] Morris-Rivera, M., Tomova, M., Wyels, C., and Yeager, A., The Radio Number of $C_n \Box C_n$, preprint available at faculty.csuci.edu/cynthia.wyels/REU/aids.htm.

[11] Ortiz, J.P., Martinez, P., Tomova, M., and Wyels, C., Radio numbers of some generalized prism graphs, preprint available at faculty.csuci.edu/cynthia.wyels/REU/aids.htm.

[12] Sooryanarayana, B., and Raghunath, P., Radio labeling of cube of a cycle, Far East J. Appl. Math. **29** (2007), no. 1, 113–147.