Progress on rainbow connection

Ingo Schiermeyer

Institut für Diskrete Mathematik und Algebra, Technische Universität Bergakademie Freiberg, 09596 Freiberg, Germany, Ingo.Schiermeyer@tu-freiberg.de

Key words: Rainbow colouring, rainbow connectivity, extremal problem

1 Introduction

We use [1] for terminology and notation not defined here and consider finite and simple graphs only.

An edge-coloured graph G is called *rainbow-connected* if any two vertices are connected by a path whose edges have different colours. This concept of rainbow connection in graphs was recently introduced by Chartrand et al. in [4]. The rainbow connection number of a connected graph G, denoted $rc(G)$, is the smallest number of colours that are needed in order to make G rainbow connected. An easy observation is that if G has n vertices then $rc(G) \leq n-1$, since one may colour the edges of a given spanning tree of G with different colours, and colour the remaining edges with one of the already used colours. Chartrand et al. computed the precise rainbow connection number of several graph classes including complete multipartite graphs [4]. The rainbow connection number has been studied for further graph classes in [3] and for graphs with fixed minimum degree in ([3], [7], [9]).

Rainbow connection has an interesting application for the secure transfer of classified information between agencies (cf. [5]). While the information needs to be protected since it relates to national security, there must also be procedures that permit access between appropriate parties. This two-fold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated). An immediate question arises: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct?

The computational complexity of rainbow connectivity has been studied in ([2], [8]). It is proved that the computation of $rc(G)$ is NP-hard ([2],[8]). In fact it is already NP-complete to decide if $rc(G) = 2$, and in fact it is already NP-complete to decide whether a given edge-coloured (with an unbounded number of colours) graph is rainbow connected [2]. More generally it has been shown in [8], that for any fixed $k \geq 2$, deciding if $rc(G) = k$ is NP-complete.

For the rainbow connection numbers of graphs the following results are known (and obvious).

Proposition 1
Let G be a connected graph of order n. Then
1. $1 \leq rc(G) \leq n-1$,
2. $rc(G) \geq diam(G)$,
3. $rc(G) = 1 \Leftrightarrow G$ *is complete,*
4. $rc(G) = n-1 \Leftrightarrow G$ *is a tree.*

2 Rainbow connection and minimum degree

Motivated by the fact that there are graphs with minimum degree 2 and with $rc(G) = n-3$ (just take two vertex-disjoint triangles and connect them by a path of length $n-5$), and by the fact that cliques have $rc(G) = 1$, it is interesting to study the behaviour of $rc(G)$ with respect to the minimum degree $\delta(G)$. In [3] Caro et al. have shown the following theorem.

Theorem 1 *If G is a connected graph with n vertices and $\delta(G) \geq 3$ then $rc(G) < \frac{5n}{6}$.*

They also made the following conjecture.

Conjecture 1 *If G is a connected graph with n vertices and $\delta(G) \geq 3$ then $rc(G) < \frac{3n}{4}$.*

For 2-connected graphs Conjecture 1 is true. This follows from the following proposition in [3].

Proposition 2 *If G is a 2-connected graph with n vertices then $rc(G) \leq \frac{2n}{3}$.*

Corollary 2 *If G is a 2-connected graph with n vertices then $rc(G) \leq \frac{3n-1}{4}$*

for $n \geq 3$.

Conjecture 1 has recently been proven in [9] by the following theorem.

Theorem 3 *If* G *is a connected graph with* n *vertices and* $\delta(G) \geq 3$ *then* $rc(G) \leq \frac{3n-1}{4}$.

The presented results motivate the following challenging problem.

Problem 1 *For every* $k \geq 2$ *find a minimal constant* c_k *with* $0 < c_k \leq 1$ *such that* $rc(G) \leq c_k \cdot n$ *for all graphs* G *with minimum degree* $\delta(G) \geq k$*. Is it true that* $c_k = \frac{3}{k+1}$ *for all* $k \geq 2$*?*

This is true for $k = 2, 3$ as shown before ($c_2 = 1$ and $c_3 = \frac{3}{4}$).

3 Rainbow connection and size of graphs

Another approach for achieving upper bounds is based on the size (number of edges) of the graph. Those type of sufficient conditions are known as Erdős-Gallai type results. Research on the following Erdős-Gallai type problem has been started in [6].

Problem 2 *For every* $k, 1 \leq k \leq n-1$, *compute and minimize the function* $f(n,k)$ *with the following property: If* $|E(G)| \geq f(n,k)$, *then* $rc(G) \leq k$.

First we can show a lower bound for $f(n,k)$.

Proposition 3
$f(n,k) \geq \binom{n-k+1}{2} + (k-1)$.

Proof: We construct a graph G_k as follows: Take a $K_{n-k+1} - e$ and denote the two vertices of degree $n-k-1$ with u_1 and u_2. Now take a path P_k with vertices labeled $w_1, w_2, \ldots, w_k$ and identify the vertices u_2 and w_1. The resulting graph G_k has order n and size $|E(G)| = \binom{n-k+1}{2} + (k-2)$. For its diameter we obtain $d(u_1, w_k) = diam(G) = k+1$. Hence $f(n,k) \geq \binom{n-k+1}{2} + (k-1)$. $\square$

Using Propositions 2 and 3 we can compute $f(n,k)$ for $k \in \{1, n-2, n-1\}$.

Proposition 4
$f(n,1) = \binom{n}{2}$,
$f(n,n-1) = n-1$,
$f(n,n-2) = n$.

For $k = 2$ we obtain $f(n, 2) = \binom{n-1}{2} + 1$ by the following stronger result shown in [6].

Theorem 4 *Let G be a connected graph of order n and size m. If $\binom{n-1}{2} + 1 \leq m \leq \binom{n}{2} - 1$, then $rc(G) = 2$.*

References

[1] J. A. Bondy and U.S.R. Murty, *Graph Theory*, Springer, 2008.

[2] S. Chakraborty, E. Fischer, A. Matsliah, and R. Yuster, *Hardness and algorithms for rainbow connectivity*, Proceedings STACS 2009, to appear in Journal of Combinatorial Optimization.

[3] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, and R. Yuster *On rainbow connection*, The Electronic Journal of Combinatorics 15 (2008), #57.

[4] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang, *Rainbow connection in graphs*, Math. Bohemica 133 (2008) 85-98.

[5] A. B. Ericksen, *A matter of security*, Graduating Engineer & Computer Careers (2007) 24-28.

[6] A. Kemnitz and I. Schiermeyer, *Graphs with rainbow connection number two*, preprint 2009, submitted to Discussiones Mathematicae Graph Theory.

[7] M. Krivelevich and R. Yuster, *The rainbow connection of a graph is (at most) reciprocal to its minimum degree*, Preprint 2009.

[8] V. B. Le and Z. Tuza, *Finding optimal rainbow connection is hard*, Preprint 2009.

[9] I. Schiermeyer, *Rainbow connection in graphs with minimum degree three*, IWOCA 2009, LNCS 5874 (2009) 432-437.