Mixed connectivity of Cartesian graph products and bundles

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1 Introduction

An interconnection network should be fault tolerant, because practical communication networks are exposed to failures of network components. Both failures of nodes and failures of connections between them happen and it is desirable that a network is robust in the sense that a limited number of failures does not break down the whole system. A lot of work has been done on various aspects of network fault tolerance, see for example the survey [6] and more recent papers [9,12,14]. In particular the fault diameter with faulty vertices which was first studied in [10] and the edge fault diameter has been determined for many important networks recently [1–4,7,8,11,13]. Usually either only edge faults or only vertex faults are considered, while the case when both edges and vertices may be faulty is studied rarely. In recent work on fault diameter of Cartesian graph products and bundles [1–4], analogous results were found for both fault diameter and edge fault diameter. However, the proofs for vertex and edge faults are independent, and our effort to see

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how results in one case may imply the others was not successful. A natural question is whether it is possible to design a uniform theory that would enable unified proofs or provide tools to translate results for one type of faults to the other. It is therefore of interest to study general relationships between invariants under simultaneous vertex and edge faults. Some basic results on edge, vertex and mixed fault diameters for general graphs appear in [5]. In order to study the fault diameters of graph products and bundles under mixed faults, it is important to understand the generalized connectivities. We define mixed connectivity which generalizes both vertex and edge connectivity, and observe some basic facts for any connected graph. Furthermore, we generalize results of vertex connectivity and edge connectivity of Cartesian graph bundles [1,4]. As a corollary, mixed connectivity of the Cartesian product of finite number of factors is given. In particular Theorem 3.2 improves the result on edge connectivity of Cartesian graph products and bundles.

2 Mixed connectivity

Definition 2.1 *Let G be any connected graph. A graph G is (p, q)+*connected*, if G remains connected after removal of any p vertices and any q edges.*

Any connected graph G is $(0, 0)$+connected, $(p, 0)$+connected for any $p < \kappa(G)$ and $(0, q)$+connected for any $q < \lambda(G)$, where $\kappa(G)$ and $\lambda(G)$ are the usual vertex- and edge-connectivities. In our notation $(i, 0)$+connected is the same as $(i + 1)$-connected, i.e. the graph remains connected after removal of any i vertices. Similarly, $(0, j)$+connected is the same as $(j + 1)$-edge connected, i.e. the graph remains connected after removal of any j edges. Clearly, if G is (p, q)+connected graph, then G is (p', q')+connected for any $p' \leq p$ and any $q' \leq q$. Furthermore, for any connected graph G with $k < \kappa(G)$ faulty vertices, at least k edges are not in functional. Roughly speaking, graph G remains connected if any faulty vertex in G is replaced with any edge. It is easy to prove that if a graph G is (p, q)+connected and $p > 0$, then G is $(p - 1, q + 1)$+connected. Hence for $p > 0$ we have a chain of implications: (p, q)+connected $\Longrightarrow$ $(p - 1, q + 1)$+connected $\Longrightarrow \ldots \Longrightarrow$ $(1, p - 1 + q)$+connected $\Longrightarrow$ $(0, p + q)$+connected, that generalizes the well-known proposition that any k-connected graph is also k-edge connected. Therefore, a graph G is (p, q)+connected if and only if $p < \kappa(G)$ and $p + q < \lambda(G)$.

If for a graph G $\kappa(G) = \lambda(G) = k$, then G is (i, j)+connected exactly when $i + j < k$. However, if $2 \leq \kappa(G) < \lambda(G)$, the question whether G is (i, j)+connected for $1 \leq i < \kappa(G) \leq i + j < \lambda(G)$ is not trivial. The example below shows that in general knowing $\kappa(G)$ and $\lambda(G)$ is not enough to decide whether G is (i, j)+connected.

Example 2.2 *For graphs on Fig. 1 we have* $\kappa(G_1) = \kappa(G_2) = 2$ *and* $\lambda(G_1) = \lambda(G_2) = 3$. *Both graphs are* $(1,0)+$*connected* $\Longrightarrow$ $(0,1)+$*connected, and* $(0,2)+$ *connected. Graph* G_1 *is not* $(1,1)+$*connected, while graph* G_2 *is.*

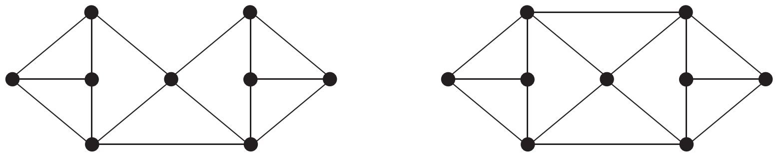

Fig. 1. Graphs G_1 and G_2 from Example 2.2.

Both edge connectivity and vertex connectivity of a graph can be computed in polyonimal time. Therefore it is interesting to ask

Problem. Let G be a graph and $1 \leq i < \kappa(G) \leq i + j < \lambda(G)$. Is there a polynomial algorithm to decide whether G is $(i, j)+$connected?

3 Mixed connectivity of Cartesian graph products and bundles

Graph products and bundles are among frequently studied interconnection network topologies. For example the meshes, tori, hypercubes and some of their generalizations are Cartesian products. It is less known that some well-known topologies are Cartesian graph bundles, i.e. some twisted hypercubes and multiplicative circulant graphs. Graph bundles also appear as computer topologies. A well known example is the twisted torus, a Cartesian graph bundle with fibre C_4 over base C_4 is the ILLIAC IV architecture, a famous supercomputer that inspired some modern multicomputer architectures. It may be interesting to note that the original design was a graph bundle with fibre C_8 over base C_8, but due to high cost a smaller version was build. A Cartesian graph bundle is a generalization of graph cover and the Cartesian graph product.

Definition 3.1 *Let* B *and* F *be graphs. A graph* G *is a* Cartesian graph bundle with fibre F over the base graph B *if there is a* graph map $p : G \to B$ *such that for each vertex* $v \in V(B)$, $p^{-1}(\{v\})$ *is isomorphic to* F, *and for each edge* $e = uv \in E(B)$, $p^{-1}(\{e\})$ *is isomorphic to* $F \square K_2$.

We have generalized the result [1] on (vertex) connectivity and improved the result [4] on edge connectivity:

Theorem 3.2 *Let* G *be a Cartesian graph bundle with fibre* F *over the base graph* B, *graph* F *be* $(p_F, q_F)+$*connected and graph* B *be* $(p_B, q_B)+$*connected. Then Cartesian graph bundle* G *is* $(p_F + p_B + 1, q_F + q_B)+$*connected.*

As the Cartesian product is a Cartesian graph bundle where all the isomorphisms between the fibres are identities, the statement about mixed connectivity of Cartesian graph products of a finite number of factors follows easily from Theorem 3.2.

Corollary 3.3 *Let graphs* $G_i, i = 1, \ldots, k$, *be* (p_i, q_i)+*connected. Then the Cartesian graph product* $G = G_1 \square G_2 \square \ldots \square G_k$ *is* $(\sum p_i + k - 1, \sum q_i)$+*connected.*

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