Pixel Guards in Polyominoes

Val Pinciu

Department of Mathematics, Southern Connecticut State University, New Haven, CT 06515

Key words: art gallery problem, pixel, polyomino

1 Introduction

The original art gallery problem, posed by Klee in 1973, asks to find the minimum number of guards sufficient to cover any polygon with n vertices. The first solution to this problem was given by Chvátal [1], who proved that $\lfloor n/3 \rfloor$ guards are sometimes necessary, and always sufficient to cover a polygon with n vertices. Later Fisk [2] provided a shorter proof of Chvátal's theorem using an elegant graph coloring argument. Klee's art gallery problem has since grown into a significant area of study. Numerous *art gallery problems* have been proposed and studied with different restrictions placed on the shape of the galleries or the powers of the guards. (See the monograph by O'Rourke [4], and the surveys by Shermer [5] and Urrutia [6].)

In this paper we consider a variation of the art gallery problem where the gallery is an *m-polyomino*, consisting of a connected union of m 1×1 unit squares called *pixels*. Throughout this paper P_m denotes an m-polyomino. We say that a point $a \in P_m$ *covers* a point $b \in P_m$ provided $a = b$, or the line segment ab does not intersect the exterior of P_m. We say that a pixel A *covers* a point b, provided some point $a \in A$ covers b. A set of points G is called a *point guard set* for P_m if for every point $b \in P_m$ there is point $a \in G$ such that a covers b. A set of pixels $\mathcal{G}$ is called a *pixel guard set* for P_m if for every point $b \in P_m$ there is a pixel $A \in \mathcal{G}$ such that A covers b.

In [3], Irfan et al. show that $\lceil \frac{m-1}{3} \rceil$ point guards are sufficient and sometimes necessary to cover any m-polyomino P_m, with $m \geq 2$. They also note that $\lceil \frac{m-1}{3} \rceil$ is an upper bound for the minimum number of pixel guards sufficient to cover any m-polyomino. In this paper we improve this bound, showing that an m-polyomino always has a pixel guard set of cardinality $\lfloor \frac{m+1}{11} \rfloor + \lfloor \frac{m+5}{11} \rfloor + \lfloor \frac{m+9}{11} \rfloor$. We also show that this bound is sharp, by constructing m-polyominoes that require exactly $\lfloor \frac{m+1}{11} \rfloor + \lfloor \frac{m+5}{11} \rfloor + \lfloor \frac{m+9}{11} \rfloor$ pixel guards.

2 Main Results

Here is our main result:

Theorem 1 *For any m-polyomino P_m with $m \geq 2$, $\lfloor \frac{m+1}{11} \rfloor + \lfloor \frac{m+5}{11} \rfloor + \lfloor \frac{m+9}{11} \rfloor$ pixel guards are always sufficient, and sometimes necessary to cover P_m.*

Proof. We will use a construction to prove the necessity part of our result. The polyomino P_{11k+2} from Figure 1 has $3+7k+4(k-1)+3 = 11k+2$ pixels. The dual graph of this polyomino is a tree with $1+2k+(k-1)+1 = 3k+1$ leaves. Since two pixels that correspond to a leaf cannot be guarded by the same pixel guard, then the number of pixels required to guard P_{11k+2} is at least $3k+1$. Simple alterations of this construction can provide examples of m-polyominos that require at least $\lfloor \frac{m+1}{11} \rfloor + \lfloor \frac{m+5}{11} \rfloor + \lfloor \frac{m+9}{11} \rfloor$ pixel guards, for any integer $m \geq 2$. Next we will prove several technical lemmas, and the sufficiency will follow from Proposition 2. □

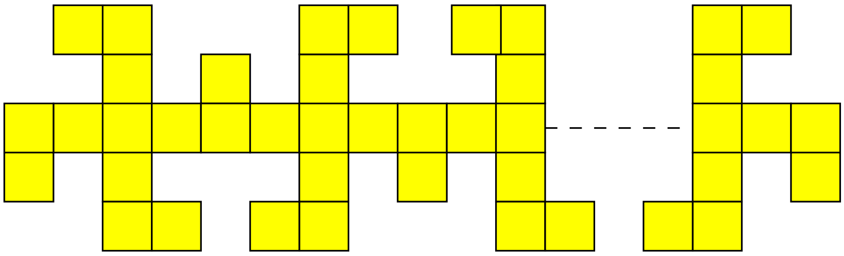

Fig. 1. An $(11k+2)$-polyomino that requires $3k+1$ pixel guards.

Lemma 1 *For each positive integer m we define*

$$f(m) = \left\lfloor \frac{m+1}{11} \right\rfloor + \left\lfloor \frac{m+5}{11} \right\rfloor + \left\lfloor \frac{m+9}{11} \right\rfloor$$

Then the following are true:

(1) $f(m+3) \leq f(m)+1 \leq f(m+4)$ *for all positive integers m.*
(2) $f(m+7) \leq f(m)+2 \leq f(m+8)$ *for all positive integers m.*
(3) $f(m+11) = f(m)+3$ *for all positive integers m.*
(4) $f(m+n-2) \geq f(m)+f(n)-1$ *for all positive integers m and n.*

Lemma 2 *For any m-polyomino P_m with $m \geq 13$, there exists a k, $4 \leq k \leq 10$ such that P_m is the union of a k-polyomino P_k and an $(m-k)$-polyomino P_{m-k}. Moreover, if the smallest k that satisfies this property is $k = 10$, then we can assume that exactly one pixel of P_{m-10} is adjacent to P_{10}.*

Proof. Given an m-polyomino P_m, let G^*_m be the dual graph of P_m, and let T_m be a spanning tree of G^*_m. Since every vertex of G^*_m has maximum degree

4, we can look at T_m as a rooted trinary tree. For simplicity, we will label the vertices of the rooted tree as the corresponding pixels in the dual polyomino. We will also transfer the common terminology from rooted trees (child, parent, sibling, etc) to the corresponding pixels. For any vertex A of T_m that is not the root, we can obtain a spanning forest of G_m^* with two components, by deleting the edge that connects A with its parent. These two componenets will generate a decomposition of P_m into two polyominoes: a k-polyomino P_k that contains A, called the *polyomino generated by A and T_m*, and another polyomino P_{m-k} that does not contain A. If B is a pixel of P_{m-k} and C is a pixel of P_k such that B and C are adjacent, we can create another spanning tree T'_m of G_m^* by replacing the edge that connects C with its parent in T_m with the edge BC. We will call this an *adoption* and say that B adopted C. An adoption will transfer a pixel of the polyomino generated by A, and all its descendants to the complementary polyomino. Now if h is the height of T_m, we consider the pixels of level $h-1$, $h-2$, $h-3$, or $h-4$ that have at least three descendants. Obviously this set is not empty. Let A be such a pixel with a minimum number of descendants. Then one can show that the polyomino generated by A satisfies the conditions of the proposition, or we can do an adoption to decrease the number of descendants of A. □

Lemma 3 *(1) One pixel guard is always sufficient to cover any 5-polyomino.*
(2) Two pixel guards are always sufficient to cover any 9-polyomino.
(3) Three pixel guards are always sufficient to cover any 12-polyomino.

Proposition 2 *For any m-polyomino P_m, if $m \geq 2$, then $\lfloor \frac{m+1}{11} \rfloor + \lfloor \frac{m+5}{11} \rfloor + \lfloor \frac{m+9}{11} \rfloor$ pixel guards are sufficient to cover P_m.*

Proof. The proof of this proposition is by induction on m. If $2 \leq m \leq 12$, the statement follows from Lemma 3. If $m \geq 13$, then by Lemma 2, P_m is the union of a k-polyomino P_k and an $(m-k)$-polyomino P_{m-k}, where $4 \leq k \leq 10$. Assume k is the smallest with this property. Let $f(m)$ be the function from Lemma 1. Then by induction hypothesis the minimum number of pixel guards required to watch P_{m-k} is $g(P_{m-k}) \leq f(m-k)$.
If $k = 4$ or $k = 5$, we obtain:
$g(P_m) \leq g(P_k) + g(P_{m-k}) \leq 1 + g(P_{m-k}) \leq 1 + f(m-4) \leq f(m)$.
If $k = 8$ or $k = 9$, we obtain:
$g(P_m) \leq g(P_k) + g(P_{m-k}) \leq 2 + g(P_{m-k}) \leq 2 + f(m-8) \leq f(m)$.

If $k = 6$, we should note that 33 out of the 35 possible hexaminoes can be covered by only one pixel guard, and we can use an argument similar with the case $k = 4$ or $k = 5$. Otherwise, if P_k requires two pixel guards, let A be the pixel that generated P_k, and let B be the parent of A. If B is the only pixel in P_{m-k} adjacent to P_k, then one can show that P_{m-k} has a pixel guard set of cardinality $f(m-4)$ that contains B. Then B can also be used to guard part of P_k, and we need only one additional guard. Otherwise, let C be a pixel

in P_{m-k} adjacent to P_k. Then C can adopt a descendant of A, reducing the problem to the case $k = 4$ or $k = 5$, or C is a leaf, in which case we can remove B and C from P_{m-k}, add them to P_k, and reduce the problem to the case $k = 8$. (note that the minimality of k was not used in the case $k = 8$.)
If $k = 7$, let A be the pixel that generated P_k, and let B be its parent. Then this case can also be reduced to the one of the cases $k = 4$, $k = 5$, or $k = 8$, or we can show that B has exactly two children. In this last case, let C be the other child of B, and let D be the parent of B. If in T_m we remove the edge BC, and the edge that connects D with its parent, we can obtain a decomposition of P_m into three polyominos. One of them is 9-polyomino. Using the induction hypothesis and Lemma 1 we obtain:
$g(P_m) \leq g(P_9) + g(P_l) + g(P_{m-l-9}) \leq 2 + f(l) + f(m - l - 9)$
$\leq 2 + f(l + m - l - 9 - 2) + 1 = 3 + f(m - 11) = f(m)$.
Finally, if $k = 10$, since k is the smallest that satisfies the property from Lemma 2, we can assume that exactly one pixel of P_{m-10} is adjacent to P_{10}. Then by removing this pixel from P_{m-10}, and adding it to P_{10}, we can assume that P_m is the union of an 11-polyomino P_{11} and an $(m-11)$-polyomino P_{m-11}. Then $g(P_m) \leq g(P_{11}) + g(P_{m-11}) \leq 3 + g(P_{m-11}) \leq 3 + f(m-11) = f(m)$. □

References

[1] V. Chvátal, A combinatorial theorem in plane geometry: *J. Combin. Theory Ser. B* **18** (1975) 39–41.

[2] S. Fisk, A short proof of Chvátal's watchman theorem: *J. Combin. Theory Ser. B* **24** (1978) 374.

[3] M. Irfan, J. Iwerks, J. Kim, J. Mitchell, Guarding Polyominoes: *19th Annual Workshop on Computational Geometry* (2009).

[4] J. O'Rourke: *Art Gallery Theorems.* Oxford University Press, 1987.

[5] T.C. Shermer: Recent results in art gallery theorems, *Proc. IEEE* **80** (1992) 1384–1399.

[6] J. Urrutia: Art gallery and illumination problems, in: *Handbook of Computational Geometry,* J.-R Sack and J. Urrutia (Eds.), Elsevier Science B. V., 1999, pp. 973–1027.