Maximum Δ-edge-colorable subgraphs of class II graphs

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1 Introduction

We consider finite, undirected graphs $G = (V, E)$ with vertex set V and edge set E. The graphs might have multiple edges but no loops. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of a graph G, respectively. A partial proper t-edge-coloring of a graph G is an assignment of colors $\{1, ..., t\}$ to some edges of G such that adjacent edges receive different colors. If θ is a partial proper t-edge-coloring of a graph G and P is a path, then P is called to be $\alpha - \beta$ alternating, if the edges of P are colored by the colors α or β. A partial proper t-edge-coloring of a graph G is called a proper t-edge-coloring (or just t-edge-coloring) of G if all edges are assigned some color. The least number t for which G has a t-edge-coloring is called the chromatic index of G and is denoted by $\chi'(G)$. The classical theorems of Shannon and Vizing state:

Theorem 1 *(Shannon) For any graph G:* $\Delta(G) \leq \chi'(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor$.

Theorem 2 *(Vizing) For any graph G:* $\Delta(G) \leq \chi'(G) \leq \Delta(G)+\mu(G)$, *where* $\mu(G)$ *is the maximum multiplicity of an edge in G.*

A graph G with $\chi'(G) = \Delta(G) = \Delta$ is *class I*, otherwise it is *class II*. There are long standing open conjectures on the class II graphs, cf. [9]. It is a notorious difficult open problem to characterize class II graphs or even to obtain some insight into their structural properties. This paper focuses on the Δ-colorable

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part of graphs. A subgraph H of G is called *maximum* Δ-edge-colorable, if it is Δ-edge-colorable and contains as many edges as possible. The fraction $|E(H)|/|E(G)|$ is subject of many papers, and e.g. lower bounds are proved for cubic, subcubic or 4-regular graphs, [1,3,5]. The aim of this paper is to prove a general best possible lower bound for all graphs.

Let H be a maximum Δ-edge-colorable subgraph of G, which is properly colored with $\{1, ..., \Delta\}$. Usually, we will refer to edges of $E(G)\backslash E(H)$ as uncolored edges. For a vertex v of G let $C(v)$ be the set of colors that appear at v, and $\bar{C}(v) = \{1, ..., \Delta\}\backslash C(v)$ be the set of colors which are missing at v. Let $e = (v, u) \in E(G)\backslash E(H)$ be an uncolored edge, and $\alpha \in \bar{C}(u)$, $\beta \in \bar{C}(v)$. Since H is a maximum Δ-edge-colorable subgraph of G, we have that $\alpha \in C(v)$ and $\beta \in C(u)$. Consider the $\alpha - \beta$ alternating path P starting from the vertex v. Again, since H is a maximum Δ-edge-colorable subgraph of G, the path P ends in u. Thus P is an even path, which together with the edge e forms an odd cycle. We will denote this cycle by $C^e_{\alpha,\beta,H}$. If the subgraph H is fixed, then we will shorten the notation to $C^e_{\alpha,\beta}$.

The cycles corresponding to uncolored edges, that are the cycles $C^e_{\alpha,\beta}$, play a central role in [6,8] in the study of cubic graphs. One aim of the present paper is to generalize some of these results to arbitrary graphs, and to investigate the maximum Δ-edge-colorable subgraphs. We show that any set of vertex disjoint cycles of a graph G with $\Delta(G) \geq 3$ can be extended to a maximum Δ-edge-colorable subgraph of G. In particular, any 2-factor of a graph with maximum degree at least three can be extended to such a subgraph.

For a graph G let $r_e(G)$ denote the minimum number of edges that should be removed from G in order to obtain a graph H with $\chi'(H) = \Delta(G)$. Let G be a graph and ϕ a $\chi'(G)$-coloring of G with $\chi'(G) = \Delta(G) + k$ $(k \geq 1)$. Let $r'_\phi(G) = \min \sum_{j=1}^{k} |\phi^{-1}(i_j)|$, and define $r'_e(G) = \min_\phi r'_\phi(G)$ as the minimum size of the union of k color-classes in a $\chi'(G)$-edge-coloring of G.

Clearly, $r_e(G) = |E(G)| - |E(H)|$, where H is a maximum Δ-edge-colorable subgraph of G. In [4] it is shown that the complement of any maximum 3-edge-colorable subgraph of a cubic graph is a matching, and hence $r_e(G) = r'_e(G)$ for cubic graphs. This paper generalizes this result to simple graphs. We further prove some bounds for the vertex degrees of a maximum $\Delta(G)$-colorable subgraph H.

2 The main results

The key property of cycles corresponding to uncolored edges that is used in [6,8] is their vertex-disjointness. There are many examples showing that they can have even common edges in the general case. Despite this, it turns out that, as Theorem 3 demonstrates below, the edge-disjointness of the cycles can be preserved.

Theorem 3 *Let H be any maximum $\Delta(G)$-edge-colorable subgraph of a graph G, and let $E(G) - E(H) = \{e_i = (u_i, v_i) | 1 \leq i \leq n\}$ be the set of uncolored edges. Assume that H is properly edge-colored with colors $1, \dots, \Delta(G)$. Then there is an assignment of colors $\alpha_1 \in \overline{C}(u_1), \beta_1 \in \overline{C}(v_1), \dots, \alpha_n \in \overline{C}(u_n), \beta_n \in \overline{C}(v_n)$ to the uncolored edges such that $E(C^{e_i}_{\alpha_i,\beta_i}) \cap E(C^{e_j}_{\alpha_j,\beta_j}) = \emptyset$, for all $1 \leq i < j \leq n$.*

The next Theorem generalizes the result of [4] that any 2-factor of a cubic graph can be extended to a maximum 3-edge-colorable subgraph to arbitrary graphs.

Theorem 4 *Let $\bar{F}$ be any set of vertex-disjoint cycles of a graph G with $\Delta = \Delta(G) \geq 3$. Then there is a maximum Δ-edge-colorable subgraph H of G, such that $E(\bar{F}) \subseteq E(H)$.*

Corollary 1 *Let $\bar{F}$ be any $2-$factor of a graph G with $\Delta(G) \geq 3$. Then there is a maximum $\Delta(G)$-edge-colorable subgraph H of G, such that $E(\bar{F}) \subseteq E(H)$.*

Let G be a graph. The length of the shortest (odd) cycle of the underlying simple graph of G is called *(odd) girth* of G. If $X \subseteq V(G)$, then $\partial_G(X)$ denotes the set of edges with precisely one end in X.

Theorem 5 *Let H be any maximum $\Delta(G)$-edge-colorable subgraph of a graph G. Then*

(1) $|\partial_H(X)| \geq \lceil \frac{|\partial_G(X)|}{2} \rceil$ *for each* $X \subseteq V(G)$.
(2) $d_H(x) \geq \lceil \frac{d_G(x)}{2} \rceil$ *for each vertex* x *of* G.
(3) $\delta(H) \geq \lceil \frac{\delta(G)}{2} \rceil$.

The bounds are best possible.

Theorem 6 *Let G be a graph with girth $g \in \{2k, 2k+1\}$ ($k \geq 1$), and H a maximum $\Delta(G)$-edge-colorable subgraph of G, then $|E(H)| \geq \frac{2k}{2k+1}|E(G)|$, and the bound is best possible.*

Theorem 7 *Every simple graph G contains a maximum Δ-edge-colorable subgraph such that the uncolored edges form a matching.*

Theorem 7 is equivalent to:

Theorem 8 *For any simple graph G: $r_e(G) = r'_e(G)$.*

It can be shown that a maximum Δ-edge-colorable subgraph of a multigraph can be class II as well. This cannot be the case for simple graphs as the following theorem shows.

Theorem 9 *Let H be a maximum $\Delta(G)$-edge-colorable subgraph of a simple graph G, then $\Delta(H) = \Delta(G)$, i.e. H is class I.*

Theorem 7 says, that every simple class II graph G has a maximum Δ-colorable subgraph H, such that $\chi'(G\backslash E(H)) = 1$. We believe that this can be generalized to multigraphs:

Conjecture 1 *Let G be a graph with $\chi'(G) = \Delta(G) + k$ $(k \geq 0)$. Then there is a maximum $\Delta(G)$-colorable subgraph H of G such that $\chi'(G\backslash E(H)) = k$.*

This conjecture is equivalent to the following statement.

Conjecture 2 *For any graph G: $r_e(G) = r'_e(G)$.*

References

[1] M. O. Albertson, R. Haas, Parsimonious edge coloring, Disc. Math. 148 (1996) 1-7

[2] H. A. Kierstead, On the chromatic index of multigraphs without large triangles, J. Comb. Theory, Ser. B 36 (1984) 156-160

[3] V. V. Mkrtchyan, S. Petrosyan, G. Vardanyan, On disjoint matchings in cubic graphs, Disc. Math. to appear, (available at: http://arxiv.org/abs/0803.0134)

[4] V. V. Mkrtchyan, S. Petrosyan, G. Vardanyan, On disjoint matchings in cubic graphs: maximum 3-edge-colorable subgraphs, under review, (available at: http://arxiv.org/abs/0909.2767)

[5] R. Rizzi, Approximating the maximum 3-edge-colorable subgraph problem, Disc. Math. 309 (2009) 4166-4170.

[6] E. Steffen, Classifications and characterizations of snarks, Disc. Math. 188 (1998) 183-203.

[7] E. Steffen, A refinement of Vizing's theorem. Disc. Math. 218 (2000) 289-291.

[8] E. Steffen, Measurements of edge-uncolorability, Disc. Math. 280 (2004) 191-214.

[9] T. R. Jensen, B. Toft, Graph coloring problems, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., New York, 1995.