Lattices and maximum flow algorithms in planar graphs

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1 Introduction

The special case of flows in planar graphs has always played a significant role in network flow theory. The predecessor of Ford and Fulkerson's well-known path augmenting algorithm – and actually the first combinatorial flow algorithm at all – was a special version for s-t-planar networks, i.e., those networks where s and t can be embedded adjacent to the infinite face [3]. The basic idea of this *uppermost path algorithm* is to iteratively augment flow along the "uppermost" non-saturated s-t-path in the planar embedding of the network. In 2006, Borradaile and Klein [1] established an intuitive generalization of this algorithm to arbitrary planar graphs, which relies on a partial order on the set of s-t-paths in the graph, called the *left/right relation*.

We connect these results from planar network flow theory with another field of combinatorial optimization, the optimization on lattice structures. In 1978, Hoffman and Schwartz introduced the notion of *lattice polyhedra* [5], a generalization of Edmond's polymatroids that is based on lattices, and proved total dual integrality of the corresponding inequaltity systems if certain additional properties hold, which are defined below.

Definition 1 *Let E be a finite set, $\mathcal{L} \subseteq 2^E$ and $\preceq$ be a partial order on $\mathcal{L}$. Then $(\mathcal{L}, \preceq)$ is a* lattice *if for any pair of elements $S, T \in \mathcal{L}$, there is a unique largest common lower bound $S \wedge T$ called* meet *and a unique least common upper bound $S \vee T$ called* join. *A lattice is* submodular *if $(S \wedge T) \cap (S \vee T) \subseteq S \cap T$ and $(S \wedge T) \cup (S \vee T) \subseteq S \cup T$ for all $S, T \in \mathcal{L}$. It is* consecutive *if $S \cap U \subseteq T$ for all $S, T, U \in \mathcal{L}$ with $S \preceq T \preceq U$.*

Based on the total dual integrality result by Hoffman and Schwartz, several different versions of two-phase-greedy algorithms were developed by Frank [4] and Faigle and Peis [2] in order to solve linear programming problems on lattice polyhedra like the packing problem

$$\max \left\{ r^T y \ : \ y \in \mathbb{R}_+^{\mathcal{L}}, \sum_{S \in \mathcal{L}: e \in S} y(S) \le c(e) \ \forall e \in E \right\}$$

and its dual if $\mathcal{L}$ is a submodular and consecutive lattice and the objective function r is supermodular and monotone [1] . Clearly, the path formulation of the maximum flow problem is a special case of the general packing problem.

Results: We show that the left/right relation induces a submodular lattice on the set of simple s-t-paths in a planar graph. If the network is s-t-planar, this lattice is also consecutive, thus meeting all requirements of Hoffman and Schwartz' framework. This implies that Ford and Fulkerson's uppermost path algorithm is a special case of the two-phase greedy algorithm on lattice polyhedra (with $r \equiv 1$). Even more, this algorithm can also solve a weighted flow problem, if the weights on the paths are supermodular and monotone. An additional result will show that whenever the graph is just planar but not s-t-planar, there is no partial order on the paths that induces a consecutive and submodular lattice.

2 The left/right relation and the path lattice

We are given a directed graph $G = (V, E)$ with a fixed planar embedding, a fixed infinite face f_∞ and two designated vertices $s, t \in V$. In our setting, paths are allowed to use edges in either direction [2] , i.e., every path P is represented by a subset of $\overleftrightarrow{E} := \{\overrightarrow{e}, \overleftarrow{e} : e \in E\}$ such that $\overrightarrow{e} \in P$ if P uses the edge e in its forward direction and $\overleftarrow{e} \in P$ if P uses e in backward direction. We denote the set of all simple s-t-paths in G by $\mathcal{P} \subseteq 2^{\overleftrightarrow{E}}$. We will analyze a partial order on $\mathcal{P}$ called left/right relation and show that it induces a submodular lattice (cf. Theorem 2), which is furthermore consecutive if the embedding is s-t-planar (cf. Theorem 3). Finally, we will show that there is no partial order on $\mathcal{P}$ that induces a consecutive and submodular lattice, if there is no s-t-planar embedding of the graph (cf. Theorem 4).

2.1 The left/right relation

The left/right relation as presented in this subsection is a partial order on $\mathcal{P}$ due to Klein [6]. We consider the *cycle space* of G, i.e., the subspace of all those vectors in $\mathbb{R}^E$ that fulfill flow conservation at all vertices. The elements of the cycle space are called *circulations*. The vectors corresponding to the clockwise

[1] These requirements on the weight function r are explained in [2].

[2] The resulting lattice can later be restricted to directed paths, maintaining all its properties.

boundary of the non-infinite faces in the embedded graph form a basis of the cycle space. Thus, for every circulation, there is a unique face potential, i.e., an assignment of numbers to the faces corresponding to the circulation. We say a circulation is *clockwise* if the corresponding face potential is non-negative.

A path $P \in \mathcal{P}$ induces a vector $\delta_P \in \mathbb{R}^E$ by $\delta_P(e) = 1$ if $\overrightarrow{e} \in P$, $\delta_P(e) = -1$ if $\overleftarrow{e} \in P$ and $\delta_P(e) = 0$ otherwise. It is easy to see that for two paths $P, Q \in \mathcal{P}$, the vector $\delta_P - \delta_Q$ is a circulation. We say that P *is left of* Q and write $P \succeq Q$ if this circulation is clockwise. It can be easily verified that the left/right relation arising from this definition is a partial order on $\mathcal{P}$. More details on the definition can be found in [1] and [7].

2.2 The path lattice in planar graphs in general

We give a short description of how to obtain a largest common lower bound of two paths $P, Q \in \mathcal{P}$ with respect to the left/right relation. Let ϕ be the face potential corresponding to the circulation $\delta_P - \delta_Q$. For $S^+ := \{f \in V^* : \phi(f) > 0\}$, we define $\delta^{P \wedge Q} := \delta_P - \sum_{f \in S^+} \phi(f)\delta_f$. The vector $\delta^{P \wedge Q}$ induces a set of darts

$$D^{P \wedge Q} := \{\overrightarrow{e} : \delta^{P \wedge Q}(e) = 1\} \cup \{\overleftarrow{e} : \delta^{P \wedge Q}(e) = -1\}.$$

Intuitively speaking, the set $D^{P \wedge Q}$ is obtained by subtracting the "clockwise part" of the circulation $P - Q$ from P. Unfortunately, this set is not necessarily a simple path. However, by flow decomposition, it can be decomposed into a path and several cycles, which can be shown to be clockwise. From this, it can be derived that the path actually is the meet of P and Q. Analogously, one can construct a set $D^{P \vee Q}$ containing the join of P and Q. A detailed proof of this result can be found in [7].

Theorem 2 $(\mathcal{P}, \preceq)$ *is a submodular lattice with $P \wedge Q$ being the unique simple s-t-path contained in $D^{P \wedge Q}$ and $P \vee Q$ being the unique simple s-t-path contained in $D^{P \vee Q}$.*

Note that the path lattice is not consecutive in general.

2.3 The path lattice in s-t-planar graphs

In case the embedding of G is s-t-planar, meet and join of the path lattice can be characterized in a more convenient way than in the general case, and even more, the lattice turns out to be consecutive. As already mentioned, Ford and Fulkerson used the existence of a unique uppermost path from s to t in every s-t-planar graph for their uppermost path algorithm. Formally, this uppermost path (and likewise a lowermost path) can be defined as the unique path with the infinite face to the left (right) of all of its elements.

For some paths $P, Q \in \mathcal{P}$ let the subgraph of G containing only the edges of P and Q be denoted by $G[E(P \cup Q)]$. It can be shown that P is left of

Q if and only if P is the uppermost path in $G[E(P \cup Q)]$ (or, equivalently, Q is the lowermost path in $G[E(P \cup Q)]$). Given this characterization of the left/right relation in s-t-planar graphs, it is easy to verify that if P and Q are incomparable, join and meet are also the uppermost and lowermost paths of $G[E(P \cup Q)]$. In order to show consecutivity, one verifies that $P \succeq Q \succeq R$ implies that P and R are uppermost and lowermost path of $G[E(P \cup Q \cup R)]$ as well, and thus any dart in $d \in P \cap R$ is a loop in the dual and, by cycle/cut duality, a one-element s-t-cut in the primal graph. This implies $d \in Q$.

Theorem 3 *If the embedding of G is s-t-planar, $(\mathcal{P}, \preceq)$ is a consecutive and submodular lattice with $P \wedge Q$ being the lowermost path in $G[E(P \cup Q)]$ and $P \vee Q$ being the uppermost path in $G[E(P \cup Q)]$.*

As a direct corollary, Ford and Fulkerson's uppermost path algorithm [3] turns out to be a special case of Phase 1 of the two-phase greedy algorithm for submodular lattice polyhedra [2].

2.4 The negative result and a characterization of s-t-planar graphs

As pointed out above, the path lattice is not consecutive on planar graphs in general. It can actually be shown that no graph that is planar but not s-t-planar can be equipped with a partial order of the paths that achieves consecutivity and submodularity at the same time. Together with the above positive result for s-t-planar graphs, we achieve the following characterization of s-t-planar graphs.

Theorem 4 *A graph is s-t-planar if and only if it is planar and there is a partial order on the set of its s-t-paths that induces a consecutive and submodular lattice.*

Sketch of proof. Consider the two graphs K_5^- and $K_{3,3}^-$ that are obtained from the Kuratowski graphs K_5 and $K_{3,3}$ by deleting the edge connecting s and t. It can be shown by very elementary arguments that for both K_5^- and $K_{3,3}^-$ there is no partial order of the s-t-paths that induces a submodular and consecutive lattice. The result then follows by applying Kuratowski's Theorem.

3 Conclusion

We provided an extensive analysis of the left/right relation on the set of s-t-paths in a planar graph. The relation induces a submodular lattice, which is even consecutive if the graph is s-t-planar. The latter result implies that the uppermost path algorithm by Ford and Fulkerson is a special case of the two-phase greedy algorithm for packing problems on submodular lattice polyhedra. We furthermore showed that submodularity and consecutivity cannot be achieved simultaneously by any partial order if the graph is not s-t-planar, thus providing a characterization of this special class of planar graphs.

References

[1] G. Borradaile, P. Klein, An $O(n \log n)$ algorithm for maximum *st*-flow in a directed planar graph, in: Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms, ACM, New York, 2006, pp. 524–533.

[2] U. Faigle, B. Peis, Two-phase greedy algorithms for some classes of combinatorial linear programs, in: Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, ACM, New York, 2008, pp. 161–166.

[3] L. R. Ford, Jr., D. R. Fulkerson, Maximal flow through a network, Canadian Journal of Mathematics. Journal Canadien de Mathématiques 8 (1956)

[4] A. Frank, Increasing the rooted-connectivity of a digraph by one, Mathematical Programming 84 (3, Ser. B) (1999) 565–576. 399–404.

[5] A. J. Hoffman, D. E. Schwartz, On lattice polyhedra, in: A. Hajnal, V. T. Sós (Eds.), Proceedings of the Fifth Hungerian Colloquium on Combinatorics, Vol. I, Vol. 18 of Colloquia mathematica Societatis János Bolyai, North-Holland, Amsterdam, 1978, pp. 593–598.

[6] P. N. Klein, Multiple-source shortest paths in planar graphs, in: Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms, ACM, New York, 2005, pp. 146–155.

[7] J. Matuschke, Lattices and maximum flow algorithms in planar graphs, Diploma thesis, TU Berlin (2009).