On Hamiltonian cycles through prescribed edges of a planar graph

Jochen Harant

Department of Mathematics, Technical University of Ilmenau, D-98684 Ilmenau, Germany

Key words: planar graph, hamiltonian cycle, prescribed edges

We use [3] for terminology and notation not defined here and consider finite simple graphs only.
The first major result on the existence of hamiltonian cycles in graphs embeddable in surfaces was by H. Whitney [12] in 1931, who proved that 4-connected maximal planar graphs are hamiltonian. In 1956, W.T. Tutte [10,11] generalized Whitney's result from maximal planar graphs to arbitrary 4-connected planar graphs. Actually, Tutte proved that a 4-connected planar graph G has a hamiltonian cycle through any two edges of a given face of G. Moreover, in [7,8] it is proved that a 4-connected planar graph G has a hamiltonian cycle through any three edges of a given face of G or that face is a 3-gon.
Improving a result of C. Thomassen [9], in 1997, D.P. Sanders [7] proved the following:

Theorem 1 *([7]) Every 4-connected planar graph on at least three vertices has a hamiltonian cycle through any two of its edges.*

In [6] the connectivity of a subset X of the vertex set of a graph G is defined as follows. Let G be a graph, $X \subseteq V(G)$, and $G[X]$ be the subgraph of G induced by X. A set $V \subset V(G)$ *splits* X if the graph $G - V$ obtained from G by removing V contains at least two components each containing a vertex of X. Let $\kappa(X)$ be infinity if $G[X]$ is complete, or the minimum cardinality of a set $V \subset V(G)$ splitting X. Let us remark that G is k-connected if and only if $\kappa(V(G)) \geq k$.

Theorem 2 is a local version of Theorem 1 and in case $X = V(G)$, Theorem 1 follows from Theorem 2. It is proven in [6].

Theorem 2 *([6]) If G is a planar graph, $X \subseteq V(G)$, $|X| \geq 3$, $\kappa(X) \geq 4$, $E \subset E(G[X])$, and $|E| \leq 2$, then G contains a cycle C with $X \subseteq V(C)$ and $E \subset E(C)$.*

The following theorem is proven in [4] and, unlike Theorem 2, it is appropriable if $|E| \geq 3$.

Theorem 3 *([4], Theorem 6) If G is a graph, $X \subseteq V(G)$, $E \neq \emptyset$ is a set of independent edges of $G[X]$, $|X| \geq 2|E| + 1$, and $\kappa(X) \geq |X| - |E|$, then G contains a cycle C with $X \subseteq V(C)$ and $E \subset E(C)$.*

Note that Theorem 3 holds for arbitrary graphs. Obviously $|X| \geq 2|E|$ since E is a set of independent edges of $G[X]$.[1] If $X = V(G)$, $|X| - |E| \geq 6$, and G is planar then Theorem 3 cannot be used since a planar graph is at most 5-connected.

We call a maximal planar graph G a plane triangulation if G is embedded into the plane. In [1] it was shown that there are 4-connected plane triangulations containing seven faces of arbitrary distance apart such that each hamiltonian cycle of that graph misses at least one of these faces, i.e. seven edges cannot be guaranteed to belong to a hamiltonian cycle of a 4-connected planar graph even if their pairwise distance is large. Theorem 1 is best possible in the sense that even three prescribed edges need not belong to a hamiltonian cycle of a 4-connected maximal planar graph. Given two edges xy and uv of a graph G, the number of edges of a shortest path in G connecting a vertex of $\{x, y\}$ and a vertex of $\{u, v\}$ is called the *distance* of xy and uv.

Theorem 4 *([5]) There is a 4-connected plane triangulation G containing $E \subseteq E(G)$ with $3|E| = |E(G)|$ such that each hamiltonian cycle of G contains exactly two edges of E. Moreover, for given integer $k \geq 1$, G and E can be chosen such that E contains three edges of pairwise distance at least k.*

The situation changes in comparison with Theorem 4, if the connectivity of G is increased and the pairwise distance of the edges in the set E is at least three. In this case it is even possible to forbid edges of E to belong to a hamiltonian cycle as described in the following Theorem 5. A proof is given in [2].

Theorem 5 *([2]) Let G be a 5-connected plane triangulation and E be a set of edges of G such that the distance between any two edges of E is at least three. Furthermore, let $E = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$. Then G has a hamiltonian cycle C with $E_1 \subset E(C)$ and $E_2 \cap E(C) = \emptyset$.*

Theorem 5 does not hold if the 5-connected plane graph G is not a triangulation. Moreover, the existence of a cycle satisfying the assertion of Theorem

[1] The inequality $|X| \geq 2|E| + 1$ is needed in the proof of Theorem 3 because of technical reasons. Probably Theorem 3 also holds in case $|X| \geq 2|E|$. If in Theorem 3, additionally, G is assumed to be planar, then it remains an open problem whether the inequality $\kappa(X) \geq |X| - |E|$ can be weakened.

5 cannot be guaranteed in the case $|E| \geq 5$ and $E_2 = \emptyset$, because for each integer k there is a 5-connected plane graph G containing a set $E \subset E(G)$ with $|E| = 5$ such that any two edges of E have distance at least k and there is no cycle of G containing E.

For three edges of a 5-connected plane triangulation the distance condition in Theorem 5 can be omitted as follows.

Theorem 6 *([5]) Let G be a 5-connected plane triangulation and E be a set of three edges of G such that E does not form a facial cycle and there is no vertex incident with all edges of E. Then G has a hamiltonian cycle containing E.*

Considering 5-connected maximal planar graphs, the following theorem is an analogue to Theorem 4.

Theorem 7 *([5]) Let G be a 5-connected plane triangulation containing an independent set V of vertices such that each face of G is incident with exactly one vertex of V and $H = G - V$ is 3-regular. Then each hamiltonian cycle of G contains exactly $\frac{1}{3}|E(H)| - 2 = \frac{1}{3}(|V(G)| - 8)$ edges of H.*

Let e_k be the smallest integer l such that there is a 5-connected plane triangulation G containing l edges of pairwise distance at least k such that there is no hamiltonian cycle of G containing all these l edges. If e_k does not exist then we write $e_k = \infty$.

Theorem 5 implies that $e_k = \infty$ if $k \geq 3$.

Theorem 8 *([5]) For e_1 the inequalities $4 \leq e_1 \leq 9$ hold.*
Moreover, there are infinitely many 5-connected maximal planar graphs G containing a set E of $\frac{1}{3}|E(G)|$ independent edges such that each hamiltonian cycle of G misses two edges of E.

It remains open whether e_2 is finite or not.

References

[1] T. Böhme, J. Harant, On Hamiltonian Cycles in 4- and 5-connected Plane Triangulations, *Discrete Mathematics*, 191(1998),25-30.

[2] T. Böhme, J. Harant, M. Tkáč, On Certain Hamiltonian Cycles in Planar Graphs, *Journal of Graph Theory* 32(1999)81-96.

[3] R. Diestel, Graph Theory, Springer, Graduate Texts in Mathematics 173(2000).

[4] T. Gerlach, J. Harant, On a Cycle through a Specified Linear Forest of a Graph, *Discrete Mathematics*, 307(2007)892-895.

[5] F. Göring, J.Harant, Hamiltonian cycles through prescribed edges of 4-connected maximal planar graphs, *Discrete Mathematics* 310(2010) 1491-1494.

[6] J. Harant, S. Senitsch, A Generalization of Tutte's Theorem on Hamiltonian Cycles in Planar Graphs, *Discrete Mathematics* 309(2009)4949-4951.

[7] D.P. Sanders, On Paths in Planar Graphs, *Journal of Graph Theory* 24(1997)341-345.

[8] R. Thomas, X. Yu, Projective Planar Graphs are Hamiltonian, *J. Combin. Theory* Ser. B 62(1994)114-132.

[9] C. Thomassen, A Theorem on Paths in Planar Graphs, *Journal of Graph Theory* 7(1983)169-176.

[10] W.T. Tutte, A Theorem on Planar Graphs, *Trans. Amer. Math. Soc.* 82(1956)99-116.

[11] W.T. Tutte, Bridges and Hamiltonian Circuits in Planar Graphs, *Aequationes Math.*15(1977)1-33.

[12] H. Whitney, A Theorem on Graphs, *Ann.Math.* 32(1931)378-390.