Chromatic index of chordless graphs

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Abstract

A graph G is said to be *chordless* if no cycle in G has a chord. Chordless graphs are exactly the graphs whose line graphs are wheel-free, which implies a connection between the study of the chromatic index of chordless graphs and the study of the chromatic number of wheel-free graphs. For example, chordless graphs of maximum degree $\Delta = 3$ are Δ-edge-colourable, and this implies the 3-vertex-colorability of {wheel,ISK_4}-free graphs [11]. In the present work we investigate the chromatic index of chordless graphs with higher degrees. We describe a decomposition result for chordless graphs and use this result to prove that every chordless graph of maximum degree $\Delta \geq 3$ has chromatic index Δ.

Key words: chordless graphs, chromatic index, edge-colouring

1 Introduction

Let $G = (V, E)$ be a simple graph. The maximum degree of a vertex in G is denoted $\Delta(G)$. A *k-edge-colouring* of G is a function $\pi : E \to \{1, 2, ..., k\}$ such that no two adjacent edges receive the same colour $c \in \{1, 2, ..., k\}$. The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-colouring. Vizing's theorem [16] states that $\chi'(G) = \Delta(G)$, and G said to be *Class 1*, or $\chi'(G) = \Delta(G) + 1$, and G said to be *Class 2*. Edge-colouring is NP-complete for regular graphs [8,10] of degree $\Delta \geq 3$. The problem is NP-complete also for the following classes [4]: comparability (hence perfect) graphs, line graphs of bipartite graphs (hence line graphs and clique graphs), {induced k-cycle}-free graphs ($k \geq 3$), cubic graphs of girth k ($k \geq 4$). Graph classes for which edge-colouring is polynomially solvable include the following: bipartite graphs [9], split-indifference graphs [12], series-parallel graphs (hence outerplanar) [9], k-outerplanar graphs [2] ($k \geq 1$). The complexity of edge-colouring is unknown for several well-studied strong structured graph classes, for which only partial results have been reported, such as cographs [1], join graphs [7], planar graphs [14], chordal graphs, and several subclasses of chordal graphs such as indifference graphs [6], split graphs [5] and interval graphs [3].

Lêvéque, Maffray and Trotignon [11] studied the class of chordless graphs, which are the graphs whose cycles are all chordless. There are two main motivations to study the class. The first motivation is its relation with the the class of wheel-free graphs: chordless graphs are exactly the graphs whose line graphs are wheel-free. Hence, the study of the chromatic index of chordless graphs has importance to the study of the (vertex) chromatic number of wheel-free graphs and subclasses. For example, the 3-vertex-colourability of {wheel, ISK_4}-free graphs is consequence [11] of the fact that chordless graphs of maximum degree 3 are Class 1. The second motivation for the study of the chromatic index of chordless graph is the fact that they are a subclass of the class of graphs that do not contain, as induced subgraph, a cycle with unique chord, called *unichord-free* graphs. The edge-colouring problem is NP-complete when restricted to unichord-fre graphs [13]; hence, it is of interest to determine subclasses of unichord-free graphs for which ege-colouring is polynomial.

2 Structure of chordless graphs

We describe a decomposition result for chordless graphs. This result is used in Section 3 to determine the chromatic index of chordless graphs.

A graph is *strongly 2-bipartite* if it is square-free and bipartite with bipartition (X, Y) where every vertex in X has degree 2 and every vertex in Y has degree at least 3. A graph is *sparse* if every edge is incident to a vertex of degree at most 2. A *cutset* of a graph G is a set of vertices whose exclusion disconnects G. The following cutsets are used in the known decomposition theorems of the class of chordless graphs:

- A *1-cutset* of a connected graph $G = (V, E)$ is a node v such that V can be partitioned into sets X, Y and $\{v\}$, so that there is no edge between X and Y. We say that (X, Y, v) is a *split* of this 1-cutset.
- A *proper 2-cutset* of a connected graph $G = (V, E)$ is a pair of non-adjacent nodes a, b, both of degree at least three, such that V can be partitioned into sets X, Y and $\{a, b\}$ so that: $|X| \geq 2$, $|Y| \geq 2$; there is no edge between X and Y, and both $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ contain an ab-path. We say that (X, Y, a, b) is a *split* of this proper 2-cutset.

A graph is called *basic* if: (1) is cycle with at least four vertices or strongly 2-bipartite; and (2) has no decomposition by 1-cutset or proper 2-cutset.

The *block* G_X (resp. G_Y) of a graph G with respect to a 1-cutset with split (X, Y, v) is $G[X \cup \{v\}]$ (resp. $G[Y \cup \{v\}]$). The *blocks* G_X and G_Y of a graph G with respect to a proper 2-cutset with split (X, Y, a, b) are defined as follows. If a node c has only neighbors a and b in G, then $G_X := G[X \cup \{a, b, c\}]$ (resp. $G_Y := G[Y \cup \{a, b, c\}]$). Otherwise, G_X (resp. G_Y) is obtained from $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$) by adding new node c adjacent to a and b.

Theorem 1 [11] describes a decomposition of chordless graphs into sparse graphs. As we state in Theorem 2, sparse graphs can be decomposed by proper 2-cutsets. From Theorems 1 and 2, we state the new decomposition result of Theorem 3. Finally, Theorem 4 proves that a non-basic chordless graph has a decomposition in which (at least) one of the blocks is basic.

Theorem 1 (Lévêque, Maffray and Trotignon [11]) *If G is a chordless graph, then either G is sparse or G admits a 1-cutset or G admits a proper 2-cutset.*

Theorem 2 *If G is sparse graph with $\Delta(G) \geq 3$ then either G is strongly 2-bipartite or G admits 1-cutset or G admits proper 2-cutset.*

Theorem 3 *If G is chordless graph then either G is strongly 2-bipartite or G is cycle on at least 4 vertices or G admits 1-cutset or G admits proper 2-cutset.*

Theorem 4 *If G is a biconnected non-basic chordless graph then G admits proper 2-cutset with split (X, Y, a, b) such that G_X is basic.*

3 Chromatic index of chordless graphs

We apply the structure results of Section 2 to show that every chordless graph of maximum degree at least 3 is Class 1. We show how to $\Delta(G)$-edge-colour a graph $G \in \mathcal{C}'$ by combining $\Delta(G)$-edge-colourings of its blocks with respect to a decomposition by proper 2-cutset.

Lemma 5 *Let G be a chordless graph of maximum degree $\Delta \geq 3$ and let (X, Y, a, b) be a split of proper 2-cutset, in such a way that G_X is basic. If G_Y is Δ-edge-colourable, then G is Δ-edge-colourable.*

Theorem 6 *Every chordless graph of maximum degree $\Delta \geq 3$ is Class 1.*

PROOF. Sketch. Assume G biconnected. If G is basic, then G is strongly 2-bipartite, hence Class 1. If G is not basic, then G has proper 2-cutset with split (X, Y, a, b) such that G_X is basic. Assume, as induction hypothesis, that G_Y is Δ-edge-colourable. By Lemma 5, graph G is Δ-edge-colourable.

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