Static symmetry breaking in circle packing

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1 Introduction

We present new Static Symmetry-Breaking Inequalities (SSBI) [11,6] for the problem of packing equal circles in a square [9]. The new SSBIs provide a marked computational improvement with respect to past work [1], though not yet at the level where a purely Mathematical Programming (MP) based spatial Branch-and-Bound (sBB) can be competitive with a Branch-and-Bound (BB) "boosted" by combinatorial and geometrical devices such as [9]. We consider the following formulation of Circle Packing in a Square (CPS) problem: given $N \in \mathbb{N}$ and $S \in \mathbb{Q}_+$, can N non-overlapping circles of unit radius be arranged in a square of side $2S$? This is equivalent to the more usual formulation where one maximizes the number of non-overlapping circles of unit radius in a square of side $2S$ with $S \in \mathbb{Q}_+$: it suffices to consider the usual correspondence (via bisection) of optimization and decision problems.

Let $\mathcal{N} = \{1, \ldots, N\}$ and $\mathcal{N}' = \{1, \ldots, N-1\}$. The CPS is formulated as the following MP:

$$\max\{\alpha \mid \forall i < j \in \mathcal{N}\ (x_i - x_j)^2 + (y_i - y_j)^2 \geq 4\alpha \wedge x, y \in [1-S, S-1]^N\} \quad (1)$$

where $(x_i, y_i) \in \mathbb{R}^2$ are the coordinates of the center of the i-th circle, for all $i \in \mathcal{N}$. For any given $N, L > 1$, if a global optimum (x^*, y^*, α^*) of (1) has $\alpha^* \geq 1$ then the CPS instance is a YES one. The CPS formulation (1) can be solved with standard off-the-shelf Mixed-Integer Nonlinear Programming (MINLP) sBB solvers such as Couenne [2]. As the instance size increases,

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these solvers yield search trees of disproportionate sizes. This is mostly due to the symmetries of the problem.

The concepts of *solution symmetries* and *formulation symmetries* were introduced in Constraint Programming [3] and brought to MP in the early 2000's [10,11]. If z is a solution of a problem P and πz is also a solution (where π permutes the components of z), π is a solution symmetry. A solution symmetry is a formulation symmetry if π also fixes the MP formulation of P. Most symmetry breaking techniques (including SSBIs) are based on formulation symmetries, because these are easier to detect. The formulation group of MINLPs (including nonconvex NLPs such as (1)) can be detected automatically using the method described in [6]. This method was shown in [7] to yield an interesting reformulation for another sphere packing problem, namely the Kissing Number Problem (KNP) [4]. Adjoining SSBIs to a formulation results in a reformulation of the *narrowing* type [5,8]: if Q is a narrowing of P then there is a mapping from the global optima $\mathcal{G}(Q)$ to the global optima $\mathcal{G}(P)$ — thus, if one is able to solve the simpler reformulation Q, then one can find a global optimum of P through the given mapping.

The automatic symmetry detection method of [6] was deployed in [1] on increasingly larger CPS instances to formulate the conjecture, and then prove, that the formulation group of the CPS is $C_2 \times S_N$, where C_2 (the cyclic group of order 2) refers to swapping x and y axes and S_N (the symmetric group of order N) refers to reindexing the circles in an arbitrary way. The constraints $\forall i \in \mathcal{N}' \ (x_i \leq x_{i+1})$ were shown in [1] to provide a narrowing of the CPS when adjoined to (1). In the rest of this paper we present a different narrowing of the CPS and discuss its impact on Couenne's performance.

2 New SSBI-based CPS narrowing

Let $L = \lfloor S \rfloor$, $\mathcal{N}'' = \{1, L+1, 2L+1, \ldots, (\lceil N/L \rceil - 2)L + 1\}$, and define the following constraint sets: $\mathscr{S} = \{x_i \leq x_{i+1} \mid i \in \mathcal{N}'\}$, $\mathscr{A}_i = \{x_h \leq x_{h+1} \mid h \in \mathcal{N}' \smallsetminus \{i + L - 1\}\}$ and $\mathscr{C}_i = \{y_i \leq y_{i+L}\}$ for all $i \in \mathcal{N}''$. Notice that these sets contain strings belonging to the formal MP language [1]: thus, when writing $\{y_i \leq y_{i+L}\}$, for example, we do not refer to the set of all points y satisfying $y_i \leq y_{i+L}$ but rather to the singleton set containing the string "$y_i \leq y_{i+L}$" as its element. Accordingly, we consider the following MP formulations: $\text{CPS}' \equiv \text{CPS} \cup \mathscr{S}$, $\text{CPS}_i \equiv \text{CPS} \cup \mathscr{A}_i \cup \mathscr{C}_i$ for all $i \in \mathcal{N}''$ and $\text{CPS}'' \equiv \text{CPS} \cup \bigcup_{i \in \mathcal{N}''} (\mathscr{A}_i \cup \mathscr{C}_i)$, where $P \cup \mathscr{D}$ denotes the MP formulation derived by adjoining constraints in $\mathscr{D}$ to P. The formulation CPS' was shown in [1] to be a narrowing of CPS.

Proposition 1 *For all $i \in \mathcal{N}''$, CPS_i is a narrowing of CPS.*

Proof. Let $i \in \mathcal{N}''$ and $(\bar{x}, \bar{y}, \bar{\alpha}) \in \mathcal{G}(\text{CPS})$. For a permutation $\pi \in S_N$ we assume $\pi(\bar{x}, \bar{y}, \bar{\alpha}) = (\pi\bar{x}, \pi\bar{y}, \bar{\alpha})$ where π acts on a vector in $\mathbb{R}^N$ by permuting the indices of its components; notice that since π is simply a reindexing of the circles, $\pi(\bar{x}, \bar{y}, \bar{\alpha}) \in$

$\mathcal{G}$(CPS). Furthermore, since CPS$'$ is known to be a narrowing of CPS, we can assume WLOG that $(\bar{x}, \bar{y}, \bar{\alpha})$ satisfies $\mathscr{S}$. If $\bar{y}_i \leq \bar{y}_{i+L}$ the result holds, otherwise assume $\bar{y}_i > \bar{y}_{i+L}$. Consider the permutation $\sigma_i = \prod_{\ell=0}^{L-1}(i+\ell, i+L+\ell)$ in S_N; $\sigma_i(\bar{x}, \bar{y}, \bar{\alpha})$ has the following properties: (a) by the action of the 2-cycle $(i, i+L)$ (appearing in σ_i when $\ell = 0$) we have $\bar{y}_i < \bar{y}_{i+L}$; (b) $\forall \ell \in \{0, \ldots, L-2\}$ we have $\sigma_i \bar{x}_{i+\ell} = \bar{x}_{i+L+\ell} \leq \bar{x}_{i+L+\ell+1} = \sigma_i \bar{x}_{i+\ell+1}$ and $\sigma_i \bar{x}_{i+L+\ell} = \bar{x}_{i+\ell} \leq \bar{x}_{i+\ell+1} = \sigma_i \bar{x}_{i+L+\ell+1}$; (c) $\forall h \in \mathcal{N}'$ such that $h \notin H_i = \{i, \ldots, i+2L-1\}$ we have $\sigma_i \bar{x}_h = \bar{x}_h \leq \bar{x}_{h+1} = \sigma_i \bar{x}_{h+1}$ because σ_i fixes all $h \notin H_i$. Thus $\sigma_i(\bar{x}, \bar{y}, \bar{\alpha}) \in \mathcal{G}(CPS)$ and satisfies the constraints of CPS$_i$. □

Lemma 2 *Let $n = \lceil N/L \rceil - 1$ and $\Sigma = \{\sigma_i \mid i \in \mathcal{N}''\}$. Then $\langle \Sigma \rangle \cong S_n$.*

Proof. Notice $\mathcal{N}'' = \{(j-1)L+1 \mid 1 \leq j \leq n\}$, and define a map $\varphi((j-1)/L+1) = j$, under which $\varphi(\Sigma) = \{(1,2), (2,3), \ldots, (n-1, n)\}$. This map induces a group homomorphism $\bar{\varphi} : \langle \Sigma \rangle \to S_n$ given by $\bar{\varphi}(\sigma_i) = (\varphi(i), \varphi(i)+1)$, which can be verified to be injective and surjective. □

Similarly, for all $h < k \in \mathcal{N}''$ we have $\langle \Sigma^{hk} \rangle = \langle \{\sigma_i \mid h \leq i < k\} \rangle \cong \mathrm{Sym}(I^{hk})$, the symmetric group on the set $I^{hk} = \{\varphi(h), \ldots, \varphi(k)\}$. Thus, for all $h, k \in \mathcal{N}''$, the permutation $\tau_{hk} = \prod_{\ell=0}^{L-1}(h+\ell, k+\ell)$ can be obtained as a certain product of the σ_i's for $i \in \varphi^{-1}(I^{hk})$. More precisely, we have $\tau_{hk} = (\varphi(k)-1, \varphi(k))(\varphi(k)-2, \varphi(k)-1) \cdots (\varphi(h), \varphi(h)+1)(\varphi(h)+1, \varphi(h)+2) \cdots (\varphi(k)-1, \varphi(k))$.

Theorem 3 *CPS$''$ is a narrowing of CPS.*

Proof. Let $(\bar{x}, \bar{y}, \bar{\alpha}) \in \mathcal{G}$(CPS), and consider the set $\mathscr{V}$ of all constraints $\mathscr{C}_i \equiv \{y_i \leq y_{i+L}\}$ violated by $(\bar{x}, \bar{y}, \bar{\alpha})$. Let ψ be the (invertible) map given by $\psi(\mathscr{C}_i) = (\varphi(i), \varphi(i)+1)$; then $\psi(\mathscr{V})$ is a set of transpositions that can be partitioned into maximal non-disjoint subsets $S^{hk} = \{(\varphi(h), \varphi(h)+1), \ldots, (\varphi(k)-1, \varphi(k))\}$; let $\mathscr{T}$ be the set of pairs (h,k) for which S^{hk} is in the partition of $\psi(\mathscr{V})$. It is easy to verify that if $\pi_{hk} = \prod_{\substack{\ell \in I^{hk} \\ h+\ell L < k-\ell L}} \tau_{h+\ell L, k-\ell L}$ then $\pi_{hk}\bar{y}$ satisfies the constraints in $\psi^{-1}(S^{hk})$. Furthermore, by maximality of the S^{hk}, the permutations π_{hk} are disjoint. Now, if $\pi = \prod_{(h,k)\in\mathscr{T}} \pi_{hk}$, $\pi(\bar{x}, \bar{y}, \bar{\alpha})$ is such that $\pi\bar{y}$ satisfies all constraints in $\mathscr{V}$ and $\pi\bar{x}$ satisfies all constraints in $\bigcup_{i\in\mathcal{N}''} \mathscr{A}_i$ by Prop. 1. Thus $\pi(\bar{x}, \bar{y}, \bar{\alpha}) \in \mathcal{G}(\mathrm{CPS}'')$. □

3 Computational results

We compare Couenne's performance on formulations CPS$'$ and CPS$''$ for some "limit" instances of CPS (i.e. N circles fit in the square but $N+1$ do not). Our comparative results, shown below, have been obtained on a 2.4GHz Intel Xeon CPU

	CPS$'$			CPS$''$		
Inst.	f^*	*nodes*	*tree*	f^*	*nodes*	*tree*
16_4	0.660	2381772	642285	**1**	2795501	839240
25_5	1	461224	188835	1	521487	222846
36_6	0	49962	23784	**1**	76409	34825
49_7	0	12577	6090	**1**	21366	10136
68_8	0	4	1	**0.943**	1057	497
86_9	0	4	1	**0.640**	5	1

with 24 GB RAM running Linux. The table displays the following statistics at termination (10h of CPU time): objective function value f^* of the incumbent, number of BB nodes closed, number of BB nodes still on the tree. The best upper bound at termination was fixed at 2 (and hence the gap was always $> 100\%$) for all reformulations and instances. However, the statistics on the number of nodes show that CPS$''$ is a better reformulation than CPS$'$. The incumbent statistics also show that CPS$''$ behaves better than CPS$'$ when used to derive heuristic solutions.

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