On finding a minimum weight cycle basis with cycles of bounded length

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1 Introduction

Consider a connected undirected graph $G = (V, E)$ without loops and multiple edges. Let $n = |V|$ and $m = |E|$ be respectively the number of vertices and edges. A generalized cycle is a subset of edges $C \subseteq E$ such that every vertex of V is incident to an even number of edges in C. All the cycles of G form a vector space, the so-called *cycle space*. Given an undirected graph G with a nonnegative weight w_e assigned to each edge $e \in E$, the *Minimum Cycle Basis* problem consists in finding a *cycle basis* $\mathcal{C}$ of minimum total weight $w(\mathcal{C}) = \sum_{C \in \mathcal{C}} w(C)$, where the weight of a cycle is defined as $w(C) = \sum_{e \in C} w_e$. This problem has been extensively studied, both from the algorithmic and the structural point of views. See the most recent works [1,2], the survey [9] and the references therein.

In this work, we investigate an interesting and natural variant, that we refer to as the *Minimum cycle basis with cycles of bounded length* problem. Given an undirected graph G with a nonnegative weight w_e and a nonnegative length l_e assigned to each edge $e \in E$ and a positive integer L, we wish to find a minimum (weight) cycle basis where each cycle C has a length $l(C) = \sum_{e \in C} l_e$ at most L. Without loss of generality we assume nonnegative integer weights and lengths on all the edges. The special case in which each cycle must contain at most k edges ($l_e = 1$ for each $e \in E$) is referred to as *Minimum k-edge-cycle basis*. Cycles with a bounded number of edges naturally arise in a number of contexts, see for instance [5], where k-edge-cycle bases play an important role.

2 Minimum cycle bases with cycles of bounded length

The existence of a cycle basis with cycles of length at most L clearly depends on the value of L and, as also noticed in [7], it can be checked in polynomial time by looking for a cycle basis with a shortest (in terms of length) longest cycle, see [3] for the algorithm. Even if the longest cycle has a length smaller than L it is hard to find one with minimum total weight.

Proposition 1 *The problem of finding a minimum cycle basis with cycles of bounded length is NP-hard.*

Proof We proceed by polynomial-time reduction from the *Partition* problem, known to be NP-complete [6], to the decision version of the above problem. In the Partition problem, given a set of N items, each with an integer size a_j, we have to decide whether there exists a subset A of items such that $\sum_{j \in A} a_j = \frac{1}{2} \sum_{j=1}^{N} a_j$. For each instance of the Partition problem it is easy to construct a special instance of the Minimum cycle basis with cycles of bounded length problem such that the answer to the former is yes if and only if the answer to the latter is yes. Consider a graph with $2N+1$ vertices $v_1 \dots v_{2N+1}$ and assume that they are ordered along a line. For every odd i, with $1 \le i \le 2N$, v_i is connected to v_{i+1} by an edge with weight $a_{(i+1)/2}$ and length 0 and to v_{i+2} by an edge with length $a_{(i+1)/2}$ and weight 0. For every even i, with $1 \le i \le 2N$, v_i is connected to v_{i+1} by an edge with both weight and length 0. The vertices v_1 and v_{2N+1} are also connected by an edge with both weight and length 0. Hence, the total number of edges is equal to $3N+1$. Let $W = L = \frac{1}{2} \sum_{j=1}^{N} a_j$. A minimum cycle basis consists of $N+1$ cycles: the N smaller cycles with both weight and length a_j (for a partial total weight $2W$) and the larger cycle given by the edge joining v_1 and v_{2N+1} plus a path through the other vertices. Finding a cycle basis with total weight $\le 3W$ and with cycles whose length is bounded by L corresponds to finding a path with total weight $\le W$ and length $\le L$ from v_1 to v_{2N+1} in the above graph without the edge joining them. This path yields a partition into nonzero-weight edges and nonzero-length edges that solves the Partition problem. $\square$

We now show that the Minimum k-edge-cycle basis problem, namely the special case where $l_e = 1$ for each $e \in E$, can be solved in polynomial time. We adapt Horton's approach [8] to the bounded problem. In Horton's algorithm, a polynomial subset of candidate cycles is generated. For every vertex $x \in V$ and every edge $e = \{y, z\} \in E$ we consider the cycle C formed by the union of the two minimum weight paths p_{xy} and p_{xz} from x to the endpoints of e, y and z, plus the edge e itself, i.e., $C = p_{xy} + p_{xz} + \{y, z\}$. We say that C has a representation $(x, \{y, z\})$. The candidate cycles are then sorted by nondecreasing weight and a minimum cycle basis is given by the $m - n + 1$ lightest independent cycles.

Unfortunately, a minimum k-edge-cycle basis is not guaranteed to be contained in the set of Horton candidate cycles. As an example, consider the sunflower graph in [9, Fig. 7] and assume that the three edges of the internal triangle have weight equal to 3 whereas the other edges have weight equal to 1. The unique 3-edge-cycle basis is given by the four triangles, but the internal one is not a Horton candidate cycle.

The proposition in [8] stating that given a cycle C in a minimum cycle basis, for any pair of vertices $u, v \in C$ the minimum weight path p_{uv} must be contained in C, is no longer valid. Denoting by p_{uv}^{l} the minimum weight path between vertices u and v with a most l edges, we have the following result.

Proposition 2 *For any two vertices u and v of a cycle C in a minimum k-edge-cycle basis, let $P_1(u,v)$ and $P_2(u,v)$ be the two paths joining vertices u and v in C. Given two integers l_1 and l_2 greater than the number of edges in $P_1(u,v)$ and $P_2(u,v)$, respectively, and such that $l_1 + l_2 = k$, at least one between $p_{uv}^{l_1}$ and $p_{uv}^{l_2}$ must be contained in C.*

Proof Suppose it is not true. Then C can be obtained as the composition of three cycles $P_1(u,v) + p_{uv}^{l_2}$, $P_2(u,v) + p_{uv}^{l_1}$, and $p_{uv}^{l_1} + p_{uv}^{l_2}$, all of lighter weight than C and with a number of edges bounded by k. Thus C cannot be contained in a minimum k-edge-cycle basis. $\square$

All the candidate k-edge-cycles can be generated by considering $C = p_{xy}^{l_1} + p_{xz}^{l_2} + \{y,z\}$ for any vertex $x \in V$, edge $e = \{y,z\} \in E$ and all the $k-2$ possible choices of positive integers l_1 and l_2 such that $l_1 + l_2 = k-1$, as in [7]. This naive approach can, however, be improved on by exploiting the notion of *isometric cycle* [1] for the unconstrained case. A cycle C is isometric if and only if it has a representation $(x, \{y,z\})$ for each vertex $x \in C$.

Proposition 3 *Every isometric cycle C has a representation $(x, \{y,z\})$ for a certain pair of vertex $x \in V$ and edge $\{y,z\} \in E$ that is balanced, i.e., such that the difference between the number of edges in p_{xy} and p_{xz} is 1 if C has an even number of edges and 0 if odd.*

For the lack of space, we cannot report the proof that is based on the efficient $O(nm)$ procedure for detecting isometric cycles proposed in [1]. The above result is also valid for cycles with at most k edges. Indeed, we only need to generate the candidate k-edge-cycles $C = p_{xy}^{l_1} + p_{xz}^{l_2} + \{y,z\}$ for every vertex $x \in V$ and edge $e = \{y,z\}$, for a choice of l_1 and l_2 leading to a balanced representation, if it exists. In this set of $O(nm)$ candidate k-edge-cycles, each cycle has a length of at most k. Since k is a constant, the total number of edges in all these cycles is $O(nm)$. Thus, by using the improved independence test recently proposed in [2], we obtain an $O(m^2 n/\log n)$ deterministic algorithm like for the unconstrained case. The independence test is inspired by de Pina's method [4], that maintains at each step a basis of the linear space that is orthogonal to the subspace spanned by the cycles selected so far. It takes advantage of the divide and conquer scheme presented in [10] and uses a bit packing technique that exploits the sparseness property, namely the fact that the number of edges in all the candidate cycles is $O(nm)$.

Finally, it is worth pointing out that, although de Pina's algorithm [4] (improved in [10]) can be easily adapted to solve Minimum k-edge-cycle basis problem by just considering minimum weight paths with at most k edges, it leads to a worse $O(m^2n + mn^2 \log n)$ complexity.

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