The classification of B-perfect graphs

Stephan Dominique Andres

Department of Mathematics and Computer Science, FernUniversität in Hagen, Germany

Key words: game chromatic number, game-perfectness, trivially perfect, graph colouring game

---

1 Introduction

Consider the following game, played on an (initially uncolored) graph $G = (V, E)$ with a color set C. The players, Alice and Bob, move alternately. A move consists in coloring a vertex $v \in V$ with a color $c \in C$ in such a way that adjacent vertices receive distinct colors. If this is not possible any more, the game ends. Alice wins if every vertex is colored in the end, otherwise Bob wins.

This type of game was introduced by Bodlaender [2]. He considers a variant, which we will call game g, in which Alice must move first and passing is not allowed. In order to obtain upper and lower bounds for a parameter associated with game g, two other variants are useful. In the game B Bob may move first. He may also miss one or several turns, but Alice must always move. In the other variant, game A, Alice may move first and miss one or several turns, but Bob must move. So in game B Bob has some advantages, whereas in game A Alice has some advantages with respect to Bodlaender's game.

For any variant $\mathcal{G} \in \{B, g, A\}$, the smallest cardinality of a color set C, so that Alice has a winning strategy for the game $\mathcal{G}$ is called *$\mathcal{G}$-game chromatic number* $\chi_{\mathcal{G}}(G)$ of G.

Email address: `dominique.andres@fernuni-hagen.de` (Stephan Dominique Andres).
URL: `http://www.fernuni-hagen.de/MATHEMATIK/DMO/mitarbeiter/andres.html` (Stephan Dominique Andres).

Let $\omega(G)$ be the clique number of a graph G. G is called *B-perfect* if, for any induced subgraph H of G, $\chi_B(H) = \omega(H)$. Analogously, we define *A-perfect* with respect to the game A and *g-perfect* with respect to Bodlaender's game. These concepts were introduced in [1] and are game-theoretic analoga of *perfect* graphs which are those graphs in which, for any induced subgraph H, the clique number equals the chromatic number $\chi(H)$. For any graph H,

$$\omega(H) \leq \chi(H) \leq \chi_A(H) \leq \chi_g(H) \leq \chi_B(H).$$

In particular, B-perfect graphs are g-perfect, g-perfect graphs are A-perfect, and A-perfect graphs are perfect. We consider the problem of characterizing these classes of graphs. The (probably most difficult) case of perfect graphs has been solved by the Strong Perfect Graph Theorem [3]:

Theorem 1 (Chudnovsky, Robertson, Seymour, Thomas (2006)) *A graph is perfect if, and only if, it does neither contain an odd hole nor an odd antihole as induced subgraph.*

In this talk we will characterize B-perfect graphs.

2 Main result

Theorem 2 *Let G be a graph. Then the following conditions are equivalent:*

(i) *G is B-perfect.*

(ii) *G does neither contain a C_4, nor a P_4, nor a split 3-star, nor a double fan as induced subgraph (see Fig. 1).*

(iii) *For every (nonempty) component H of G, there is $k \geq 0$, so that*

$$H = K_1 \vee (H_0 \cup H_1 \cup \ldots \cup H_k),$$

where the H_i are complete graphs for $i \geq 1$, and H_0 is either empty or there are $p, q, r \in \mathbb{N}$, so that $H_0 = K_p \vee K_r \vee K_q$ (see Fig. 2).

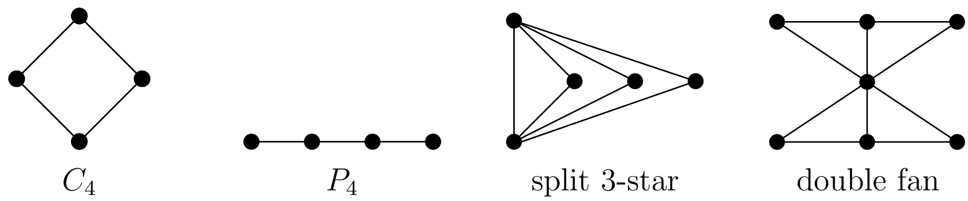

Fig. 1. 4 forbidden induced subgraphs for B-perfect graphs

PROOF. (i) $\Longrightarrow$ (ii): Winning strategies for Bob with ≤ 2 colors on C_4 resp. P_4 resp. with ≤ 3 colors on the split 3-star resp. the double fan are obvious.

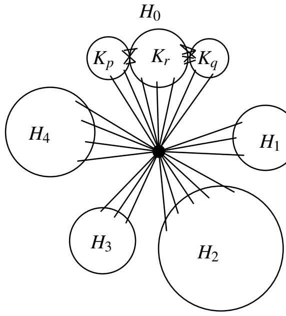

Fig. 2. Structure of a component according to (iii)

(iii) $\Longrightarrow$ (i): We describe a winning strategy for Alice with $\omega(G)$ colors on a graph G as in (iii). This is sufficient since every induced subgraph of G is of the same type as described in (iii). For $H_0 = K_p \vee K_r \vee K_q$ let the K_p and the K_q be the *ears*. Alice always responds to Bob's moves in the same component H (if Bob passes, in an arbitrary component). As long as Bob does not play in an ear, Alice does not play in an ear; she first colors the universal vertex of H. If Bob plays in an ear K_p, Alice colors a vertex in the corresponding ear K_q with the same color (in case there is no uncolored vertex she uses the strategy described before). If Alice is forced to start coloring an ear, then all non-ear-vertices are colored, so a coloring of the ears is possible without creating danger for a non-ear-vertex.

(ii) $\Longrightarrow$ (iii): We examine the structure of a graph G without induced P_4, C_4, split 3-star, double fan. Let H be a component of G. We use the following lemma of Wolk [5].

Lemma 3 (Wolk (1965)) *A connected graph without induced C_4 and P_4 (a so-called* trivially perfect graph [4]*) has a universal vertex.*

So, H has a universal vertex v. Let $H_0, \ldots, H_n$ be the components of $H \setminus v$. Using the fact that H does not contain a double fan we can prove the following

Claim 4 *At most one of the H_i is not complete.*

Let H_0 be the (only) component of $H \setminus v$ which is not complete. Let K be the largest clique of H_0. We are done if we show:

Claim 5 *$H_0 \setminus K$ induces a clique.*

Claim 6 *$H_0 \setminus K$ induces a module of H_0 (i.e. if $x \in K$, either x is adjacent to all $y \in H_0 \setminus K$ or to none.)*

The proof of Claim 5 uses Lemma 3 again and the fact that H does neither contain a split 3-star nor a P_4. The proof of Claim 6 uses Claim 5 and the fact that H does neither contain a P_4 nor a C_4.

Claim 4, Claim 5 and Claim 6 together imply that H has the structure as described in (iii): $H_0 \setminus K$ corresponds to the K_p, its neighbors correspond to the K_r, and the rest of H_0 corresponds to the K_q. This completes the proof of Theorem 2. $\square$

3 Open problems

Problem 7 *Characterize A-perfect graphs by forbidden induced subgraphs.*

Problem 8 *Characterize g-perfect graphs by forbidden induced subgraphs.*

We discuss some partial results concerning these problems. The following are already known, cf. [1]:

Theorem 9 *A triangle-free graph G is A-perfect if, and only if, every component of G is either K_1 or $K_{m,n}$ or $K_{m,n} - e$, where e is an edge.*

Theorem 10 *Complements of bipartite graphs are A-perfect.*

References

[1] Andres, S. D., *Game-perfect graphs*, Math. Meth. Oper. Res. **69** (2009), 235–250

[2] Bodlaender, H. L., *On the complexity of some coloring games*, Int. J. Found. Comput. Sci. **2**, no.2 (1991), 133–147

[3] Chudnovsky, M., N. Robertson, P. Seymour, and R. Thomas, *The strong perfect graph theorem*, Ann. Math. **164** (2006), 51–229

[4] Golumbic, M. C., *Trivially perfect graphs*, Discrete Math. **24** (1978), 105–107

[5] Wolk, E. S., *A note on "the comparability graph of a tree"*, Proc. Am. Math. Soc. **16** (1965), 17–20