Key words:

1 Introduction

Some combinatorial results have an easy proof via generating functions and a more elusive, but also more interesting and important, bijective proof. It would be difficult to think of a better example of this than the generalization of Euler's classical distinct/odd theorem due to George Andrews (Theorem 1). The proof via generating functions is a trivial one-line calculation. On the other hand, the simplest bijective proof of this result, O'Hara's algorithm, is distinctly non-trivial and has numerous fascinating properties.

Note that a quest to find bijective proofs of partition identities goes back all to way to the pioneer work of Sylvester and his school. Despite remarkable successes in the last century (see [**?**]) and some recent work of both positive and negative nature (see e.g. [**?**,**?**]), the problem remains ambiguous and largely unresolved. Much of this stems from the lack of clarity as to what exactly constitutes a bijective proof. Depending on whether one accentuates simplicity, ability to generalize, the time complexity, geometric structure, or asymptotic stability, different answers tend to emerge.

In one direction, the subject of partition bijections was revolutionized by Garsia and Milne with their *involution principle* [**?**,**?**]. This is a combinatorial construction which allows to use a few basic bijections and involutions to build more involved combinatorial maps. As a consequence, one can start with a reasonable analytic proof of a partition identity and trace every step to obtain a (possibly extremely complicated) bijective construction. Garsia and Milne used this route to obtain a long sought bijection proving the Rogers-Ramanujan identities, resolving an old problem in this sense [**?**]. Unfortunately, this bijection is too complex to be analyzed and has yet to lead to new Rogers-Ramanujan type partition identities.

After Garsia-Milne paper, there has been a flurry of activity to obtain synthetic bijections for large classes of partition identities. Most of these bijections did not seem to lead anywhere with one notable exception. Remmel and Gordon found (rather involved) bijective proofs of the above-mentioned partition identity due to Andrews [**?**,**?**]. O'Hara's streamlined proof is in fact a direct generalization of Glaisher's classical bijection proving Euler's theorem. Moreover, in her thesis [**?**], O'Hara showed that her bijection is computationally efficient in certain special cases. Until now, the reason why O'Hara's bijection has a number of nice properties distinguishing it from the other "involution principle bijections" remained mysterious.

In this extended abstract, we present results of both positive and negative type. First, we analyze the complexity of O'Hara's bijection, which we view as a discrete algorithm. Theorem 3 gives an *exact* formula for the number of steps of the algorithm in certain cases. From here it follows that O'Hara's bijection is computationally efficient in many special cases. On the other hand, perhaps surprisingly, the number of steps can be (mildly) exponential in the worst case (Theorem 7 part (3)). This is the first negative result of this kind, proving the analogue of a conjecture that remains open for the Garsia-Milne's "Rogers-Ramanujan bijection" (see Subsection **??**).

Second, we show that O'Hara's bijection has a rich underlying geometry. In a manner similar to that in [**?**,**?**], we view this bijection as a map between integer points in polytopes which preserves certain linear functionals. We present an advanced generalization of Andrews's result and of O'Hara's bijection in this geometric setting. In a special case, the working of the map corresponds to the Euclid algorithm and, more generally, to terms in the continuing fractions. Thus one can also think of our generalization as a version of multidimensional continuing fractions.

Finally, by combining the geometric and complexity ideas we see that in the finite dimensional case the map defined by O'Hara's bijection is a solution of an integer linear programming problem. This implies that the map defined by the bijection can be computed in polynomial time, i.e. much more efficiently than by O'Hara's bijection.

The extended abstract is structured as follows. We start with definitions and notations in Section 2. In Section 3, we describe the main results on both geometry and complexity. We conclude with final remarks in Section **??**.

Due to space constraints, we present almost no proofs. An interested reader is invited to find the proofs and some other results in the paper [**?**], on which this abstract is based.

2 Definitions and background

2.1 Andrews's theorem

A *partition* λ is an integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$, where the integers λ_i are called the *parts* of the partition. The sum $n = \sum_{i=1}^{\ell} \lambda_i$ is called the *size* of λ, denoted $|\lambda|$; in this case we say that λ is a partition of n, and write $\lambda \vdash n$. We can also write $\lambda = 1^{m_1} 2^{m_2} \cdots$, where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i. The *support* of $\lambda = 1^{m_1} 2^{m_2} \cdots$ is the set $\{i \colon m_i > 0\}$. The set of all positive integers will be denoted by $\mathbb{P}$.

Denote the set of all partitions by $\mathcal{P}$ and the set of all partitions of n by $\mathcal{P}_n$. The number of partitions of n is given by Euler's formula

$$\sum_{\lambda \in \mathcal{P}} t^{|\lambda|} = \sum_{n=0}^{\infty} |\mathcal{P}_n| t^n = \prod_{i=1}^{\infty} \frac{1}{1 - t^i}.$$

For a sequence $\overline{a} = (a_1, a_2, \ldots)$ with $a_i \in \mathbb{P} \cup \{\infty\}$, define $\mathcal{A}$ to be the set of partitions λ with $m_i(\lambda) < a_i$ for all i; write $\mathcal{A}_n = \mathcal{A} \cap \mathcal{P}_n$. Denote by $\operatorname{supp}(\overline{a}) = \{i \colon a_i < \infty\}$ the *support* of the sequence $\overline{a}$.

Let $\overline{a} = (a_1, a_2, \ldots)$ and $\overline{b} = (b_1, b_2, \ldots)$. We say that $\overline{a}$ and $\overline{b}$ are *φ-equivalent*, $\overline{a} \sim_\varphi \overline{b}$, if φ is a bijection $\operatorname{supp}(\overline{a}) \to \operatorname{supp}(\overline{b})$ such that $i a_i = \varphi(i) b_{\varphi(i)}$ for all i. If $\overline{a} \sim_\varphi \overline{b}$ for some φ, we say that $\overline{a}$ and $\overline{b}$ are equivalent, and write $\overline{a} \sim \overline{b}$.

Theorem 1 (Andrews) *If $\overline{a} \sim \overline{b}$, then $|\mathcal{A}_n| = |\mathcal{B}_n|$ for all n.*

Proof: We use the notation $t^\infty = 0$. Clearly,

$$\sum_{n=0}^{\infty} |\mathcal{A}_n| t^n = \prod_{i=1}^{\infty} \frac{1 - t^{i a_i}}{1 - t^i} = \prod_{j=1}^{\infty} \frac{1 - t^{j b_j}}{1 - t^j} = \sum_{n=0}^{\infty} |\mathcal{B}_n| t^n,$$

which means that $|\mathcal{A}_n| = |\mathcal{B}_n|$. $\square$

Consider the classical Euler's theorem on partitions into distinct and odd parts. For $\overline{a} = (2, 2, \ldots)$ and $\overline{b} = (\infty, 1, \infty, 1, \ldots)$, $\mathcal{A}_n$ is the set of all partitions of n into distinct parts, and $\mathcal{B}_n$ is the set of partitions of n into odd parts. The bijection $i \mapsto 2i$ between $\operatorname{supp}(\overline{a}) = \mathbb{P}$ and $\operatorname{supp}(\overline{b}) = 2\mathbb{P}$ satisfies $i a_i = \varphi(i) b_{\varphi(i)}$, so $\overline{a} \sim_\varphi \overline{b}$ and $|\mathcal{A}_n| = |\mathcal{B}_n|$. We refer to this example as the *distinct/odd case.*

2.2 O'Hara's algorithm

The analytic proof of Andrews's theorem shown above does not give an explicit bijection $\mathcal{A}_n \to \mathcal{B}_n$. Such a bijection is, by Theorem 2, given by the following algorithm.

Algorithm 1 (O'Hara's algorithm on partitions)

```
Fix: sequences ā ~φ b̄
Input: λ ∈ A
Set: μ ← λ
While: μ contains more than b_j copies of j for some j
  Do: remove b_j copies of j from μ, add a_i copies of i to μ, where φ(i) = j
Output: ψ(λ) ← μ
```

Theorem 2 (O'Hara) *Algorithm 1 stops after a finite number of steps. The resulting partition $\psi(\lambda) \in \mathcal{B}$ is independent of the order of the parts removed and defines a size-preserving bijection $\mathcal{A} \to \mathcal{B}$.*

Denote by $L_\varphi(\lambda)$ the number of steps O'Hara's algorithm takes to compute $\psi(\lambda)$, and by $\mathcal{L}_\varphi(n)$ the maximum value of $L_\varphi(\lambda)$ over all $\lambda \vdash n$.

In the distinct/odd case, O'Hara's algorithm gives the inverse of Glaisher's bijection, which maps $\lambda = 1^{m_1}3^{m_3}\cdots \in \mathcal{B}$ to the partition $\mu \in \mathcal{A}$ which contains $i2^j$ if and only if m_i has a 1 in the j-th position when written in binary. □

Let $\overline{a} = (1,1,4,5,3,1,1,\ldots)$, $\overline{b} = (1,1,5,3,4,1,1,\ldots)$ and $\varphi(3) = 4$, $\varphi(4) = 5$, $\varphi(5) = 3$, $\varphi(i) = i$ for $i \neq 3,4,5$; observe that $\overline{a} \sim_\varphi \overline{b}$. Then O'Hara's algorithm on $\lambda = 3^3 4^4 5^2$ runs as follows:

$$\mathbf{3^3 4^4 5^2} \longrightarrow 3^7 4^1 5^2 \longrightarrow 3^2 4^1 5^5 \longrightarrow 3^2 4^6 5^1 \longrightarrow 3^6 4^3 5^1$$
$$\longrightarrow 3^{10} 4^0 5^1 \longrightarrow 3^5 4^0 5^4 \longrightarrow 3^0 4^0 5^7 \longrightarrow 3^0 4^5 5^3 \longrightarrow \mathbf{3^4 4^2 5^3}$$

We have $L_\varphi(\lambda) = \mathcal{L}_\varphi(35) = 9$. □

Take $\overline{a} = (2,2,1,2,2,1,\ldots)$ and $\overline{b} = (3,1,3,1,\ldots)$. Here $\mathcal{A}$ is the set of partitions into distinct parts $\equiv \pm 1 \bmod 3$, and $\mathcal{B}$ is the set of partitions into odd parts, none appearing more than twice. Define $\varphi\colon \mathbb{P} \to \mathbb{P}$ as follows:

$$\varphi(i) = \begin{cases} i & \text{if } i \text{ is divisible by } 6 \\ i/3 & \text{if } i \text{ is divisible by 3, but not by 2} \\ 2i & \text{if } i \text{ is not divisible by 3} \end{cases} . \qquad (1)$$

Clearly, $\overline{a} \sim_\varphi \overline{b}$. O'Hara's algorithm on $1^1 2^1 8^1 10^1 14^1 20^1$ runs as follows:

$$
\begin{array}{ccccccccc}
 & \mathbf{1^1 2^1 8^1 10^1 14^1 20^1} & \to & 1^1 2^1 8^1 10^3 14^1 & \to & 1^1 2^1 7^2 8^1 10^3 & \to & 1^1 2^1 5^2 7^2 8^1 10^2 \\
\to & 1^1 2^1 5^4 7^2 8^1 10^1 & \to & 1^1 2^1 5^6 7^2 8^1 & \to & 1^1 2^1 4^2 5^6 7^2 & \to & 1^1 2^3 4^1 5^6 7^2 \\
\to & 1^1 2^5 5^6 7^2 & \to & 1^3 2^4 5^6 7^2 & \to & 1^5 2^3 5^6 7^2 & \to & 1^7 2^2 5^6 7^2 \\
\to & 1^9 2^1 5^6 7^2 & \to & 1^{11} 5^6 7^2 & \to & 1^{11} 5^3 7^2 15^1 & \to & 1^{11} 7^2 15^2 \\
\to & 1^8 3^1 7^2 15^2 & \to & 1^5 3^2 7^2 15^2 & \to & 1^2 3^3 7^2 15^2 & \to & \mathbf{1^2 7^2 9^1 15^2}
\end{array}
$$

The bijection ψ is similar in spirit to Glaisher's bijection: given $\lambda = 1^{m_1} 2^{m_2} 4^{m_4} 5^{m_5} \cdots \in \mathcal{A}$ and $j \in \mathbb{P}$, the number of copies of part $2j-1$ in $\psi(\lambda)$ is equal to the k-th digit in the ternary expansion of l, where k is the highest power of 3 dividing $2j-1$, $2j-1 = 3^k r$, and $l = \sum_i 2^i m_{r2^i}$. $\square$

2.3 Equivalent sequences and graphs

Choose equivalent sequences $\overline{a}$, $\overline{b}$. Define a directed graph G_φ on $\operatorname{supp}(\overline{a}) \cup \operatorname{supp}(\overline{b})$ by drawing an edge from i to j if $\varphi(j) = i$; an arrow from i to j therefore means that O'Hara's algorithm simultaneously removes copies of i and adds copies of j. Each vertex v has $\operatorname{indeg} v \leq 1$, $\operatorname{outdeg} v \leq 1$ and $\operatorname{indeg} v + \operatorname{outdeg} v \geq 1$. The graph splits into connected components of the following five types:

(i) cycles of length $m \geq 1$;
(ii) paths of length $m \geq 2$;
(iii) infinite paths with an ending point, but without a starting point;
(iv) infinite paths with a starting point, but without an ending point;
(v) infinite paths without a starting point or an ending point.

Figure 1 shows portions of graphs G_φ for certain φ:

(1) $\overline{a} = (1,1,4,5,3,1,1,\ldots)$, $\overline{b} = (1,1,5,3,4,1,1,\ldots)$, $\varphi(3) = 4$, $\varphi(4) = 5$, $\varphi(5) = 3$, $\varphi(i) = i$ for $i \neq 3,4,5$; components of G_φ are of type (i);
(2) $\overline{a} = (\infty,1,2,3,\infty,\infty,\infty,\ldots)$, $\overline{b} = (2,3,4,\infty,\infty,\infty,\infty,\ldots)$, $\varphi(2) = 1$, $\varphi(3) = 2$, $\varphi(4) = 3$; G_φ is of type (ii);
(3) the distinct/odd case: $\overline{a} = (2,2,\ldots)$, $\overline{b} = (\infty,1,\infty,1,\ldots)$, $\varphi(i) = 2i$; components of G_φ are of type (iii);
(4) the odd/distinct case: $\overline{a} = (\infty,1,\infty,1,\ldots)$, $\overline{b} = (2,2,\ldots)$, $\varphi(i) = i/2$; components of G_φ are of type (iv);
(5) $\overline{a} = (2,2,1,2,2,1,\ldots)$ and $\overline{b} = (3,1,3,1,\ldots)$, φ given by (1); components of G_φ are of types (i) and (v). $\square$

Fig. 1. Examples of graphs G_φ.

2.4 Scissor-congruence and Π*-congruence*

We say that convex polytopes A, B in $\mathbb{R}^m$ are *congruent*, write $A \simeq B$, if B can be obtained from A by rotation and translation. For convex polytopes $P, Q \subset \mathbb{R}^m$, we say that they are *scissor-congruent* if P can be cut into finitely many polytopes which can be rearranged and assembled into Q, i.e. if P and Q are the disjoint union of congruent polytopes: $P = \cup_{i=1}^n P_i$, $Q = \cup_{i=1}^n Q_i$, $P_i \simeq Q_i$.

Let π be a linear functional on $\mathbb{R}^m$. If Q_i can be obtained from P_i by a translation by a vector in the hyperplane $\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^m \colon \pi(\mathbf{x}) = 0\}$, we say that P and Q are π*-congruent*. If P and Q are π-congruent for some linear functional π, we say that they are Π*-congruent*.

If P can be cut into countably many polytopes which can be translated by a vector in the hyperplane $\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^m \colon \pi(\mathbf{x}) = 0\}$ and assembled into Q, we say that P and Q are *approximately* π*-congruent*. We say that they are *approximately* Π*-congruent* if they are approximately π-congruent for some linear functional π. If P and Q are approximately π-congruent, there exist, for every $\varepsilon > 0$, π-congruent polytopes $P_\varepsilon \subseteq P$ and $Q_\varepsilon \subseteq Q$, such that $\mathrm{vol}(P \setminus P_\varepsilon) < \varepsilon$ and $\mathrm{vol}(Q \setminus Q_\varepsilon) < \varepsilon$.

Finally, let $\mathbf{R}(a_1, \ldots, a_m) = [0, a_1) \times \cdots \times [0, a_m)$ be a box in $\mathbb{R}^m$, and let $R(a_1, \ldots, a_m) = \mathbf{R}(a_1, \ldots, a_m) \cap \mathbb{Z}^m$ be the set of its integer points.

Let $d = 2$ and $\pi(x, y) = x + y$. Euclid's algorithm on (a, b) yields a π-congruence between $\mathbf{R}(a, b)$ and $\mathbf{R}(b, a)$: if $b = r_1 a + s_1$ with $0 \leq s_1 < a$, divide $[0, a) \times [0, r_1 a)$ into r_1 squares with side a, and translate the square $[0, a) \times [ia, (i + 1)a)$ by the vector $(ia, -ia)$ to $[ia, (i + 1)a) \times [0, a)$. Then write $a = r_2 s_1 + s_2$ with $0 \leq s_2 < s_1$, divide $[0, a) \times [r_1 a, b)$ into r_2 squares with side s_1, and translate the square $[is_1, (i + 1)s_1) \times [r_1 a, b)$ by the vector $(r_1 a - is_1, is_1 - r_1 a)$ to $[r_1 a, b) \times [is_1, (i + 1)s_1)$. Continue until the remainder s_i is equal to 0. The first drawing of Figure 2 gives an example.

The second drawing shows that boxes $\mathbf{R}(12, 8)$ and $\mathbf{R}(32, 3)$ are π-congruent for $\pi(x, y) = x + 4y$. Finally, in Figure 3 we give a π-congruence between $\mathbf{R}(4, 5, 3)$ and $\mathbf{R}(5, 3, 4)$ for $\pi(x, y, z) = 3x + 4y + 5z$. $\square$

Fig. 2. Two Π-congruences.

$$\xrightarrow{\boldsymbol{\psi}}$$

Fig. 3. π-congruence between $\mathbf{R}(4,5,3)$ and $\mathbf{R}(5,3,4)$.

3 Main results

3.1 Continuous O'Hara's algorithm and Π*-congruences*

Take the case when G_φ is a cycle $i_1 \to i_m \to i_{m-1} \to \ldots \to i_1$. In this case, $\varphi(i_1) = i_2$, $\varphi(i_2) = i_3$, etc. Throughout this section, identify a partition $i_1^{t_1} \cdots i_m^{t_m}$ with the vector $\mathbf{t} = (t_1, \ldots, t_m)$. By Theorem 2, O'Hara's algorithm defines a bijection $\psi \colon R(a_1, \ldots, a_m) \to R(b_1, \ldots, b_m)$, where $i_j a_j = i_{j+1} b_{j+1}$ for all j, where the indices are taken cyclically. The following algorithm (see also Theorem 3) generalizes ψ to the continuous setting. It gives a bijection $\boldsymbol{\psi} \colon \mathbf{R}(a_1, \ldots, a_m) \to \mathbf{R}(b_1, \ldots, b_m)$, which is defined also for non-integer a_j, b_j. When a_j, b_j are integers, it is an extension of $\psi \colon R(a_1, \ldots, a_m) \to R(b_1, \ldots, b_m)$. As an immediate corollary, we prove that two boxes with rational coordinates and with equal volume are Π-congruent. We can use Theorem 3 to give an alternative proof of Theorem 2.

Algorithm 2 (continuous O'Hara's algorithm)

`Fix:` $\mathbf{i} = (i_1, \ldots, i_m) \in \mathbb{R}_+^m$
$\quad\quad \mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}_+^m$, $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{R}_+^m$ with $i_j a_j = i_{j+1} b_{j+1}$
`Input:` $\mathbf{t} \in \mathbf{R}(a_1, \ldots, a_m)$
`Set:` $\mathbf{s} \leftarrow \mathbf{t}$
`While:` $\mathbf{s}$ contains a coordinate $s_j \geq b_j$
$\quad$ `Do:` $s_j \leftarrow s_j - b_j$, $s_{j-1} \leftarrow s_{j-1} + a_{j-1}$
`Output:` $\boldsymbol{\psi}(\mathbf{t}) \leftarrow \mathbf{s}$

It is clear that the algorithm starts with an element of $P = \mathbf{R}(a_1, \ldots, a_m)$ and, if the while loop terminates, outputs an element of $Q = \mathbf{R}(b_1, \ldots, b_m)$. It is not obvious, however, that the loop terminates in every case, or that the output $\boldsymbol{\psi}(\mathbf{t})$ and the number of steps $\mathbf{L}_\varphi(\mathbf{t})$ depend only on $\mathbf{t}$, not on the choices made in the while loop.

Theorem 3 *Algorithm 2 has the following properties.*

(1) The algorithm stops after a finite number of steps, and the resulting vector $\boldsymbol{\psi}(\mathbf{t})$ and the number of steps $\mathbf{L}_\varphi(\mathbf{t})$ are independent of the choices made during the execution of the algorithm.
(2) The algorithm defines a bijection $\boldsymbol{\psi} \colon P \to Q$ which satisfies $\boldsymbol{\psi}(\mathbf{t}) - \mathbf{t} \in \mathcal{H}$, where $\mathcal{H}$ is the hyperplane defined by $i_1 x_1 + \ldots + i_m x_m = 0$.

(3) We have

$$\mathbf{L}_\varphi(\mathbf{t}+\mathbf{t}') \geq \mathbf{L}_\varphi(\mathbf{t}) + \mathbf{L}_\varphi(\mathbf{t}') \text{ for every } \mathbf{t}, \mathbf{t}', \mathbf{t}+\mathbf{t}' \in P.$$

In particular, $\mathbf{L}_\varphi(\mathbf{t}') \leq \mathbf{L}_\varphi(\mathbf{t})$ *if* $\mathbf{t}' \leq \mathbf{t}$.

(4) Let $\mathbf{t}, \mathbf{t}' \in P$, $\mathbf{s} = \boldsymbol{\psi}(\mathbf{t})$, *with* $t_j \leq t'_j < t_j + \varepsilon_j$, *where* $\varepsilon_j = b_j - s_j$. *Then*

$$\boldsymbol{\psi}(\mathbf{t}') - \mathbf{t}' = \boldsymbol{\psi}(\mathbf{t}) - \mathbf{t} \quad \text{and} \quad \mathbf{L}_\varphi(\mathbf{t}') = \mathbf{L}_\varphi(\mathbf{t}).$$

(5) For all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^m$, *we have*

$$\max_{\mathbf{t}\in P} \mathbf{L}_\varphi(\mathbf{t}) \;=\; \mathrm{lcm}(c_1, \ldots, c_m) \cdot \left(\frac{1}{c_1} + \ldots + \frac{1}{c_m}\right) - m,$$

where $c_j = a_1 \cdots a_{j-1} b_j \cdots b_{m-1}$.

We call boxes $P = \mathbf{R}(a_1, \ldots, a_m)$, $Q = \mathbf{R}(b_1, \ldots, b_m)$ *relatively rational* if there exists λ, $\lambda \neq 0$, such that $\lambda a_j \in \mathbb{Z}, \lambda b_j \in \mathbb{Z}$. Clearly, two boxes P and Q with rational side-lengths are relatively rational.

Corollary 4 *Boxes* $P = \mathbf{R}(a_1, \ldots, a_m)$, $Q = \mathbf{R}(b_1, \ldots, b_m)$ *with equal volume are approximately* Π*-congruent. Moreover, when* P *and* Q *are relatively rational and have equal volume, they are* Π*-congruent.*

For $j = 1, \ldots, m$, take $i_j = a_1 \cdots a_{j-1} b_{j+1} \cdots b_m$. Clearly $i_j a_j = i_{j+1} b_{j+1}$ for $j = 1, \ldots, m-1$, and $a_1 \cdots a_m = b_1 \cdots b_m$ implies $i_m a_m = i_1 b_1$. Therefore, the numbers i_j, a_j, b_j satisfy the conditions of Algorithm 2. By Theorem 3 part (2), the algorithm defines a bijection $\boldsymbol{\psi}\colon P \to Q$. Parts (4) and (2) of Theorem 3 imply that we can cut P into (countably many) smaller boxes, each of which is translated by a vector in the plane $i_1 x_1 + \ldots + i_m x_m = 0$.

If P and Q are relatively rational, we can assume without loss of generality that all a_j, b_j are integers. For any integer vector $\mathbf{t}$, we have $\boldsymbol{\psi}(\mathbf{t}') - \mathbf{t}' = \boldsymbol{\psi}(\mathbf{t}) - \mathbf{t}$ and $\mathbf{L}_\varphi(\mathbf{t}') = \mathbf{L}_\varphi(\mathbf{t})$ whenever $t_j \leq t'_j < t_j + 1$, so P and Q are divided into a finite number (at most $a_1 \cdots a_m$) of boxes.

Even in the 3-dimensional case the Π-congruence defined by the algorithm can be quite complex, as the next figure suggests. Here the same shading is used for parallel translations by the same vector. $\square$

Fig. 4. The decomposition of the box $\mathbf{R}(31, 47, 23)$ given by O'Hara's algorithm (only the top, right, and back sides are shown) .

3.2 *Complexity of O'Hara's algorithm*

The complexity of O'Hara's algorithm has been an open problem, with the exception of the elementary distinct/odd case (see [**?**]).

It turns out that the complexity depends heavily on the type of the graph G_φ defined in Subsection 2.3. Part (5) of Theorem 3 gives the maximum number of steps that O'Hara's algorithm takes when G_φ is a cycle. The following lemma gives an estimate for $\mathcal{L}_\varphi(n)$ when G_φ is a path.

Lemma 5 *Let G_φ be a finite or infinite path on $\mathcal{I} \subseteq \mathbb{P}$. Then $\mathcal{L}_\varphi(n) \leq n(\log n + 1)$. Moreover, if*

$$D = \sum_{i \in \mathcal{I}} \frac{1}{i a_i} = \sum_{j \in \mathcal{I}} \frac{1}{j b_j} < \infty,$$

then $\mathcal{L}_\varphi(n) \leq Dn$. Here, by $\log n$ we mean the natural logarithm of n.

Theorem 6 *Let $\overline{a}, \overline{b}$ be φ-equivalent sequences.*

(1) If G_φ has only a finite number of cycles of length > 2, then $\mathcal{L}_\varphi(n) = O(n \log n)$, and the constants implied by the O-notation are universal.
(2) If G_φ has only a finite number of cycles of length $> m$ for some $m > 2$, then $\mathcal{L}_\varphi(n) = O(n^{m-1})$, and the constants implied by the O-notation depend only on m.

The following theorem gives the corresponding lower bound on the worst case complexity. It shows that the estimates of Theorem 6 are close to being sharp.

Theorem 7 *There exist φ-equivalent sequences $\overline{a}$ and $\overline{b}$, such that:*

(1) G_φ is a path and $\mathcal{L}_\varphi(n) = \Omega(n \log \log n)$;
(2) G_φ contains only cycles of length $\leq m$ and $\mathcal{L}_\varphi(n) = \Omega(n^{m-1-\varepsilon})$ for every $\varepsilon > 0$;
(3) $\mathcal{L}_\varphi(n) = \exp \Omega(\sqrt[3]{n})$.

In other words, depending on the type of the graph, we have nearly matching upper and lower bounds on $\mathcal{L}_\varphi(n)$. For example, for an m-cycle, Theorem 6 shows that $\mathcal{L}_\varphi(n)$ is $O(n^{m-1})$, while Theorem 7 shows that it is $\Omega(n^{m-1-\varepsilon})$ for every $\varepsilon > 0$. Similarly, part (3) shows that O'Hara's algorithm can be very slow in general since the total number of partitions of n is asymptotically $\exp \Theta(\sqrt{n})$.

3.3 O'Hara's algorithm as an integer linear programming problem

Let us now give a new description of O'Hara's algorithm.

Proposition 8 *Let* $\mathbf{i}, \mathbf{a}, \mathbf{b} \in$ *be as above such that* $i_j a_j = i_{j+1} b_{j+1}$ *for* $j = 1, \ldots, m$. *Fix a vector* $\mathbf{t} \in \mathbf{R}(a_1, \ldots, a_m)$. *Then* $\mathbf{s} = \boldsymbol{\psi}(\mathbf{t})$ *satisfies the following:*

$$\mathbf{s} = \mathbf{t} + A\mathbf{k},$$

where

$$A = \begin{pmatrix} -b_1 & a_1 & 0 & \cdots & 0 \\ 0 & -b_2 & a_2 & \cdots & 0 \\ 0 & 0 & -b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_m & 0 & 0 & \cdots & -b_m \end{pmatrix}$$

and $\mathbf{k} = (k_1, \ldots, k_m)$ *is the unique vector minimizing*

$$k_1 + \ldots + k_m$$

with constraints

$$\mathbf{k} \in \mathbb{Z}^m, \qquad \mathbf{k} \geq \mathbf{0}, \qquad A\mathbf{k} \geq -\mathbf{t}, \qquad A\mathbf{k} \leq \mathbf{b} - \mathbf{1} - \mathbf{t}.$$

Proposition **??** can be used to obtain a significant speed-up of (the usual) O'Hara's algorithm, in the case when G_φ contains only cycles of bounded length. Namely, we obtain the following result.

Theorem 9 *Let* $\overline{a} \sim_\varphi \overline{b}$. *If the lengths of cycles of* G_φ *are bounded, there exists a deterministic algorithm which computes* $\psi(\lambda)$ *in* $O(n \log n)$ *steps for* $\lambda \in \mathcal{A}_n$.

Without loss of generality, the support of $\lambda \in \mathcal{A}_n$ is contained in one of the connected components of G_φ. If this connected component is a path, O'Hara's algorithm takes $O(n \log n)$ steps by Lemma 5. If it is a cycle of length m, we can use the algorithm described in, say, [**?**, Corollary 18.7b] to compute $\psi(\lambda)$ in $O(\log^c n)$ steps for some c. Obviously the $O(n \log n)$ term dominates.

4 Final remarks

4.1

The polynomial time algorithm in the proof of Theorem **??** is given implicitly, by using the general results in integer linear programming. It is saying that the function $\psi : \mathcal{A}_n \to \mathcal{B}_n$ can be computed much faster, by circumventing the elegant construction of O'Hara's algorithm. It would be interesting to give an explicit construction of such an algorithm.

In a different direction, it might prove useful to restate other involution principle bijections in the language of linear programming, such as the Rogers-Ramanujan bijection in [**?**] or in [**?**]. If this works, this might lead to a new type of a bijection between these two classes of partitions. Alternatively, this might resolve the conjecture by the second author on the mildly exponential complexity of Garsia-Milne's Rogers-Ramanujan bijection, see [**?**, Conjecture 8.5].

4.2

Note the gap between the number $\exp \Theta(\sqrt{n})$ of partitions of n and the lower bound $\mathcal{L}_\varphi(n) = \exp \Omega(\sqrt[3]{n})$ in Theorem 7. It would be interesting to decide which of the two worst complexity bounds on the number of steps of O'Hara's algorithm is closer to the truth.

Note that we applied our linear programming approach only in the bounded cycle case. We do not know if there is a way to apply the same technique to the general case. However, we believe that there are number theoretic obstacles preventing that and in fact, computing O'Hara's bijection as a function on partitions may be hard in the formal complexity sense.

4.3

Recently, variations on the O'Hara's bijection and applications of rewrite systems were found in [**?**] and [**?**,**?**]. It would be interesting to see connections between our analysis and this work.

4.4

Recall also that the 2-dimensional case can be viewed as the Euclid algorithm which in turn corresponds to the usual continued fractions (see Example **??**). Thus the geometry of ψ can be viewed as a delicate multidimensional extension of continued fractions. Given the wide variety of (different) multidimensional continued fractions available in the literature, it would be interesting to see if there is a connection to at least one of these notions.

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